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Laurent Baratchart, Juliette Leblond, Stéphane Rigat, Emmanuel Russ

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Laurent Baratchart * Juliette Leblond †
Stéphane Rigat ‡ Emmanuel Russ §

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*INRIA Sophia Antipolis Méditerranée, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis, France. Email: Laurent.Baratchart@sophia.inria.fr

†INRIA Sophia Antipolis Méditerranée. Email: Juliette.Leblond@sophia.inria.fr

‡Université de Provence, CMI, 39 rue F. Joliot-Curie, F-13453 Marseille Cedex 13, France, and LATP, CNRS, UMR 6632. Email: rigat@cmi.univ-mrs.fr

§Université Paul Cézanne, Faculté des Sciences et Techniques de Saint-Jérôme, Avenue Escadrille Normandie-Niémen, 13397 MARSEILLE Cedex 20, France, and LATP, CNRS, UMR 6632. E-mail: emmanuel.russ@univ-cezanne.fr

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Abstract. We study Hardy spaces of solutions to the conjugate Beltrami equation with Lipschitz coefficient on Dini-smooth simply connected planar domains, in the range of exponents $1 < p < \infty$. We analyse their boundary behaviour and certain density properties of their traces. We derive on the way an analog of the Fatou theorem for the Dirichlet and Neumann problems associated with the equation $\mathrm{div}(\sigma\nabla u) = 0$ with L^p-boundary data.
Keywords: Hardy spaces, conjugate Beltrami equation, trace, non-tangential maximal function.

1 Background and motivation

Classical Hardy spaces of the disk or the half-plane lie at the crossroads between complex and Fourier analysis, and many developments in spectral theory and harmonic analysis originate in them [53, 64, 65].
From a spectral-theoretic point of view, the shift operator and its various compressions play a fundamental role and stress deep connections between function theory on the one hand, control, approximation, and prediction theory on the other hand [54, 55, 57]. In particular, Hankel and Toeplitz operators on Hardy classes team up with standard functional analytic tools to solve extremal problems where a function, given on part or all of the boundary, is to be approximated by traces of analytic or meromorphic functions [1, 12, 13, 15, 16, 19, 27, 48, 59]. Such techniques are of particular relevance to identification and design of linear control systems [4, 17, 31, 37, 48, 57, 63]. In recent years, on regarding the Laplace equation as a compatibility condition for the Cauchy-Riemann system, analogous extremal problems were set up to handle inverse Dirichlet-Neumann issues for 2-D harmonic functions [11, 14, 18, 45, 46]. Laying grounds for a similar approach to inverse problems involving more general diffusion elliptic equations in the plane has been the initial motivation for the authors to undertake the present study. The equations we have in mind are of the form

$$\mathrm{div}(\sigma\nabla u) = 0, \qquad \sigma \text{ real-valued}, \quad 0 < c < \sigma < C, \tag{1}$$

which may be viewed, upon setting $\nu = (1-\sigma)/(1+\sigma)$, as a compatibility condition for the *conjugate* Beltrami equation

$$\overline{\partial} f = \nu \, \overline{\partial f}, \qquad f = u + iv, \qquad \nu \in \mathbb{R}, \qquad |\nu| < \kappa < 1. \tag{2}$$

This connection between (1) and (2) was instrumental in [7] for the solution of Calderón's conjecture in dimension 2. Equation (2) breaks up into a system of two real equations that reduces to the Cauchy-Riemann system when $\sigma = 1$. It differs considerably from the better known Beltrami equation: $\overline{\partial} f = \nu \, \partial f$ whose solutions are the so-called quasi-regular mappings, which are complex linear and have been extensively studied by many authors, see [3, 5, 6, 22, 28, 29, 40, 42, 43, 44, 49, 52].
An interesting example of a free boundary inverse problem involving an equation like (1) –albeit in doubly connected geometry– arises when trying to recover the surface of the plasma in plane sections of a toroidal tokamak from the so-called Grad-Shafranov equation [21].
In the present paper, we limit ourselves to the case when σ is Lipschitz-continuous. Moreover, we merely consider analogs H^p_ν to the classical Hardy spaces H^p in the range $1 < p < \infty$, on Dini-smooth simply connected domains. From the perspective of harmonic analysis, the main features of Hardy space theory in this range of exponents [32, 35] are perhaps the Fatou theorem on non-tangential boundary values, the p-summability of the non-tangential maximal function, the boundedness of the conjugation operator, which is the prototype of a convolution operator of Calderón-Zygmund type, and the fact that subsets of positive measure of the boundary are uniqueness sets (this is false in dimension greater than 2 [23]). In this work, we show that Hardy solutions to (2) enjoy similar properties, and we use them to establish the density of the traces of such solutions in $L^p(\Gamma)$ whenever Γ is a subset of non-full measure of the boundary. This fact, whose proof is straightforward for classical Hardy spaces [13] and can be generalized to harmonic gradients in higher dimensions when Γ is closed [9], is of fundamental importance in extremal problems with incomplete boundary data and one of the main outcome of the paper.
The generalized Hilbert transform $\mathcal{H}_\nu$ involved in (2), that maps the boundary values of u to the boundary values of v, was introduced and studied in [7, 8] when $p = 2$ for less smooth (*i.e.* measurable bounded) σ but smoother (*i.e.* Sobolev $W^{1/2,2}$) arguments. Here, we shall prove its L^p and $W^{1,p}$ boundedness and compare it to the classical conjugation operator. In addition, studying its adjoint will lead us to a representation theorem for the dual of H^p_ν which generalizes the classical one. The latter is again of much importance when studying extremal problems.
On our way to the proof of the density theorem, we establish regularity results for the solutions of (1) which are not entirely classical. For example, we obtain an analog of the Fatou theorem concerning solutions of the Dirichlet problem for (1) with L^p boundary data, including L^p-estimates for the nontangential maximal function. Also, the gradient of a solution to the Neumann problem with L^p data has L^p nontangential boundary values a.e. as well as L^p-summable nontangential maximal function. For the ordinary Laplacian this is known to hold on C^1 domains in all dimensions [33], and on Lipschitz domains for restricted range of p [47]. But for diffusion equations of the form (1), the authors could not locate such a result in the literature, even in dimension two; when $p = 2$ and σ is smooth, it follows from [50, Ch. II, Thms 7.3, 8.1] that this gradient converges radially in L^2 on (parallel transportations of) the boundary, and the result could be carried over

to any $p \in (1, \infty)$ using the methods of [39], but no pointwise estimates are obtained this way[1].

The definition of generalized Hardy spaces that we use –see (10) below– dwells on the existence of harmonic majorants for $|f|^p$, and also on the boundedness of L^p norms of f on Jordan curves tending to the boundary of the domain [32]. As in the classical case, these two definitions of Hardy spaces coincide on Dini-smooth domains (the only case of study below) but not over non-smooth domains –where arclength on the boundary and harmonic measure are no longer mutually absolutely continuous.

Although equation (2) is real linear only, our methods of investigation rely on complex analytic tools. In particular, we elaborate on ideas and techniques from [20] and we use standard facts from classical Hardy space theory together with well-known properties of the Beurling transform. This entails that higher dimensional analogs of our results, if true at all, require new ideas to be proven.

We made no attempt at expounding the limiting cases $p = 1, \infty$. These have generated the deepest developments in the classical theory, centering around BMO and Fefferman duality, but trying to generalize them would have made the paper unbalanced and they are left here for further research.

Finally, we did not consider Hardy spaces over doubly connected domains, in spite of the fact that the above-mentioned application to free boundary problems in plasma control takes place in an annular geometry. Including these would have made for a lengthy paper, but the results below lay ground for such a study.

2 Notations for function spaces

Throughout, $\mathbb{D}$ is the open unit disk and $\mathbb{T}$ the unit circle in the complex plane $\mathbb{C}$. We let D_r and $\mathbb{T}_r$ stand for the open disk and the circle centered at 0 with radius r. For I an open subset of $\mathbb{T}$, endowed with its natural differentiable structure, we put $\mathcal{D}(I)$ for the space of C^∞ complex functions supported on I.

If $\Omega \subset \mathbb{C}$ is a smooth domain (the meaning of "smooth" will be clear from the context), we say that a sequence $\xi_n \in \Omega$ approaches $\xi \in \partial\Omega$ non tangentially if it converges to ξ while no limit point of $(\xi_n - \xi)/|\xi_n - \xi|$ is tangent to $\partial\Omega$ at ξ. A function f on Ω has non tangential limit ℓ at ξ if $f(\xi_n)$ tends to ℓ for any sequence ξ_n which approaches ξ non tangentially.

2.1 Hölder spaces

If $\Omega \subset \mathbb{R}^2$ is open, $C^{k,\gamma}(\overline{\Omega})$ indicates the subspace of complex functions whose derivatives are bounded and continuous up to order k, while those of order k satisfy a Hölder condition of exponent $\gamma \in (0, 1]$. Such functions extend continuously to $\overline{\Omega}$ together with their derivatives of order at most k. A complete norm on $C^{k,\gamma}(\overline{\Omega})$ is obtained by putting

$$\|f\|_{C^{k,\gamma}(\overline{\Omega})} := \sup_{0\leq|\lambda|\leq k} \|f^{(\lambda)}\|_{L^\infty(\Omega)} + \sup_{\substack{|\lambda|=k \\ \xi\neq\zeta}} \frac{|f^{(\lambda)}(\xi) - f^{(\lambda)}(\zeta)|}{|\xi - \zeta|^\gamma},$$

[1]In other respects the results of [50] are of course much more general since they deal with arbitrary non-homogeneous elliptic equations in any dimension and can handle distributional boundary conditions.

where $\lambda = (\lambda_1, \lambda_2)$ is a multi-index, $|\lambda| = \lambda_1 + \lambda_2$, and $f^{(\lambda)}$ is the corresponding derivative. The space $C^\infty(\overline{\Omega}) = \cap_k C^{k,1}(\overline{\Omega})$ of smooth functions up to the boundary of Ω is topologized with the seminorms $\|\ \|_{C^{k,1}(\overline{\Omega})}$, where k ranges over $\mathbb{N}$. We put $C^{k,\gamma}_{loc}(\Omega)$ for the functions whose restriction to any relatively compact open subset Ω_0 of Ω lies in $C^{k,\gamma}(\overline{\Omega_0})$. A family of semi-norms making $C^{k,\gamma}_{loc}(\Omega)$ into a Fréchet space is given by $\|\ \|_{C^{k,\gamma}(\Omega_n)}$, with Ω_n a sequence of relatively compact open subsets exhausting Ω.

2.2 Lebesgue and Sobolev spaces

We coordinatize $\mathbb{R}^2 \simeq \mathbb{C}$ by $\xi = x + iy$ and denote interchangeably the (differential of) planar Lebesgue measure by

$$dm(\xi) = dx\,dy = (i/2) d\xi \wedge d\overline{\xi}\,,$$

where $d\xi = dx + idy$ and $d\overline{\xi} = dx - idy$. For $1 \leq p \leq +\infty$ and E a measurable subset of $\mathbb{C}$, we put $L^p(E)$ for the familiar Lebesgue space with respect to dm.
Let $\mathcal{D}(\Omega)$ be the space of complex C^∞ functions with compact support in Ω, endowed with the inductive topology. Its dual $\mathcal{D}'(\Omega)$ is the usual space of distributions on Ω. Whenever $T \in \mathcal{D}'(\Omega)$, we use the standard notations:

$$\partial T = \partial_z T = \frac{1}{2}\left(\partial_x - i\partial_y\right) T \quad \text{and} \quad \overline{\partial} T = \partial_{\overline{z}} T = \frac{1}{2}\left(\partial_x + i\partial_y\right) T,$$

and record the obvious identity: $\overline{\partial T} = \bar{\partial}\,\overline{T}$.
We denote by $W^{1,p}(\Omega)$ the Sobolev space of those $f \in L^p(\Omega)$ whose distributional derivatives ∂f and $\overline{\partial} f$ also belong to $L^p(\Omega)$. The norm on $W^{1,p}(\Omega)$ is defined by

$$\|f\|^p_{W^{1,p}(\Omega)} := \|f\|^p_{L^p(\Omega)} + \|\partial f\|^p_{L^p(\Omega)} + \|\overline{\partial} f\|^p_{L^p(\Omega)}.$$

When Ω is smooth, any function $f \in W^{1,p}(\Omega)$ has a trace on $\partial\Omega$ (all the domains under consideration in the present paper are smooth enough for this to be true) which will be denoted by $\operatorname{tr} f$.
The symbol $W^{1,p}_{loc}(\Omega)$ refers to those distributions whose restriction to any relatively compact open subset Ω_0 of Ω lies in $W^{1,p}(\Omega_0)$. Equipped with the semi-norms $\|\ \|_{W^{1,p}(\Omega_n)}$, $W^{1,p}_{loc}(\Omega)$ is a Fréchet space.
The space $W^{1,\infty}(\Omega)$ is isomorphic to $C^{0,1}(\overline{\Omega})$ [64, Ch.VI, Sec. 6.2]. In particular, every $f \in W^{1,\infty}(\Omega)$ extends (Lipschitz) continuously to the boundary $\partial\Omega$ of Ω.
A C^1-smooth Jordan curve is the injective image of $\mathbb{T}$ under a nonsingular continuously differentiable map from $\mathbb{T}$ into $\mathbb{C}$. For Γ an open subset of such a curve, we denote by $L^p(\Gamma)$ the Lebesgue space with respect to (normalized) arclength (there should be no confusion with our previous notation $L^p(E)$, as the context will always remain clear) and by $W^{1,p}(\Gamma)$ the Sobolev space of those absolutely continuous $\varphi \in L^p(\Gamma)$ whose tangential derivative $\partial_t \varphi$ with respect to arclength again lies in $L^p(\Gamma)$. A complete norm on $W^{1,p}(\Gamma)$ is given by

$$\|\varphi\|^p_{W^{1,p}(\Gamma)} := \|\varphi\|^p_{L^p(\Gamma)} + \|\partial_t \varphi\|^p_{L^p(\Gamma)}.$$

Note that, on $\mathbb{T}$, the tangential derivative $\partial_t h$ coincides with $\partial_\theta h/(2\pi)$ where $\partial_\theta h$ indicates the derivative with respect to θ when φ is written as a function of $e^{i\theta}$.

We shall have an occasion to deal with $W^{2,p}(\Omega)$, comprised of $W^{1,p}(\Omega)$-functions whose first derivatives again lie in $W^{1,p}(\Omega)$. A norm on $W^{2,p}(\Omega)$ is obtained by setting

$$\|f\|^p_{W^{2,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \|\partial f\|^p_{W^{1,p}(\Omega)} + \|\overline{\partial} f\|^p_{W^{1,p}(\Omega)}.$$

As is customary, we indicate with a subscript "0" the closure of C^∞ compactly supported functions in an ambient space.

Finally, we indicate with a subscript "$\mathbb{R}$", like in $L^p_{\mathbb{R}}(\Omega)$, the real subspace of real-valued functions in a given space.

3 Definition of Hardy spaces

3.1 An elliptic equation

In the present paper, we investigate the L^p boundary behaviour of solutions to a second order elliptic equation in divergence form on a planar domain. More precisely, let $\Omega \subset \mathbb{R}^2$ be a smooth simply connected domain (most of the time, we will take $\Omega = \mathbb{D}$, except in Section 6, where Ω will be assumed to be Dini-smooth) and $\sigma \in W^{1,\infty}(\Omega)$ be such that, for two constants c, C, one has

$$0 < c \leq \sigma \leq C\,. \tag{3}$$

With the standard notation $\nabla u := (\partial_x u\,,\, \partial_y u)^T$ and div $g = \partial_x g + \partial_y g$, where the superscript "T" means "transpose", the elliptic equation that we will consider is

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{a.e. in } \Omega, \tag{4}$$

Our approach to (4) proceeds *via* the study of a *complex* elliptic equation of the first order, namely the *conjugate* Beltrami equation:

$$\overline{\partial} f = \nu\, \overline{\partial f} \quad \text{a.e. in } \Omega\,, \tag{5}$$

where $\nu \in W^{1,\infty}(\Omega)$ is a *real valued* function that satisfies:

$$\|\nu\|_{L^\infty(\Omega)} \leq \kappa \quad \text{for some } \kappa \in (0,1)\,. \tag{6}$$

Formally, equation (5) decomposes into a system of two real elliptic equations of the second order in divergence form. Indeed, for $f = u + iv$ a solution to (5) with real-valued u, v, we see on putting $\sigma = (1-\nu)/(1+\nu)$ that u satisfies equation (4) while v satisfies

$$\operatorname{div}\left(\frac{1}{\sigma}\nabla v\right) = 0 \;\text{ a.e. in } \Omega. \tag{7}$$

Note also, from the definition of σ, that (6) implies (3). Conversely, let u be a real-valued solution to (4). Then, since $\partial_y(-\sigma\partial_y u) = \partial_x(\sigma \partial_x u)$ and Ω is simply connected, there is a real-valued function v, such that

$$\begin{cases} \partial_x v = -\sigma \partial_y u, \\ \partial_y v = \sigma \partial_x u, \end{cases} \tag{8}$$

hence $f = u + iv$ satisfies (5) with $\nu = (1-\sigma)/(1+\sigma)$. Moreover, (3) implies (6).

In the present work, we consider several classes of solutions to (5) for which the formal manipulations above will be given a precise meaning. All classes we shall deal with are embedded in $L^p(\Omega)$ for some $p \in (1,\infty)$, in which case the solutions to (4), (5), and (7) can be understood in the distributional sense. This only requires defining distributions like $\sigma\partial_x u$, which is done naturally using Leibniz's rule[2] when $\sigma \in W^{1,\infty}(\Omega)$ and $u \in L^p(\Omega)$ [28].
It will turn out that our solutions actually lie in $W^{1,p}_{loc}(\Omega)$ for some $p \in (1,\infty)$, in which case (5) may as well be interpreted in the pointwise sense while (4) becomes equivalent to

$$\int_\Omega \sigma \nabla u . \nabla g \, dm = 0, \qquad g \in \mathcal{D}_\mathbb{R}(\Omega), \tag{9}$$

where the dot indicates the Euclidean scalar product in $\mathbb{R}^2$. This follows easily from the fact that the product of a function in $W^{1,\infty}(\Omega)$ by a function in $W^{1,p}_{loc}(\Omega)$ again lies in $W^{1,p}_{loc}(\Omega)$ and its distributional derivative can be computed according to the Leibniz rule. We shall make use of these observations without further notice.

To find u with prescribed trace on $\partial\Omega$ is known as the Dirichlet problem for (4) in Ω. In light of the previous discussion, we slightly abuse terminology and still refer to the issue of finding f with prescribed $\text{Re}f$ on $\partial\Omega$ as being the Dirichlet problem for (5).

For simplicity, we shall work entirely over the unit disk $\mathbb{D}$ and only later, in Section 6, shall we indicate how one can carry our results over to Dini-smooth domains. As became customary in analysis, we tend to use the same symbol to mean possibly different constants, with subscripts indicating the dependence of the constant under examination.

When $\nu \in W^{1,\infty}(\mathbb{D})$, the solvability in $W^{1,p}(\mathbb{D})$ of the Dirichlet problem for (5) with boundary data in the fractional Sobolev space $W^{1-1/p,p}_\mathbb{R}(\mathbb{T})$ (an intrinsic definition of which can be found in [2, Thm 7.48]) is a straightforward consequence of the known solvability of the corresponding Dirichlet problem for equation (4) [24]. We shall however state and establish this fact which is our point of departure (see Theorem 4.1.1 in Section 4 below).

Below, we relax the assumptions on the boundary data, assuming only they belong to $L^p(\mathbb{T})$. Of course, the solution of the Dirichlet problem will no longer belong to $W^{1,p}(\mathbb{D})$ in general, but rather to some generalized Hardy space $H^p_\nu(\mathbb{D})$ that we shall define and study throughout the paper, starting in the next section.

3.2 Definition of Hardy spaces

For $1 < p < \infty$, we denote by $H^p(\mathbb{D})$ the classical Hardy space of holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^p(\mathbb{D})} := \operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} < +\infty \,, \tag{10}$$

[2]For instance if $f \in L^p(\Omega)$ and $\nu \in W^{1,\infty}_\mathbb{R}(\Omega)$, we define by $\nu\overline{\partial f}$ to be the distribution

$$\langle \nu\overline{\partial f}, \phi\rangle = -\int_\Omega (\nu\overline{f}\,\overline{\partial}\phi + \overline{\partial}\nu\overline{f}\phi) dm \,, \quad \forall \phi \in \mathcal{D}(\Omega) \,.$$

where

$$\|f\|_{L^p(\mathbb{T}_r)} := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{1/p},$$

by our convention that arclength gets normalized, see [32, 35].
Of course H^p can be introduced for $p = 1, \infty$ as well, but we do not consider such exponents here. We extend the previous definition to two classes of generalized analytic functions as follows.

3.2.1 The class $H^p_\nu(\mathbb{D})$

If $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ satisfies (6), and $1 < p < +\infty$, we define a generalized Hardy space $H^p_\nu(\mathbb{D})$ to consist of those Lebesgue measurable functions f on $\mathbb{D}$ satisfying

$$\|f\|_{H^p_\nu(\mathbb{D})} := \operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} < +\infty \tag{11}$$

that solve (5) in the sense of distributions on $\mathbb{D}$; note that (11) implies $f \in L^p(\mathbb{D})$. It is not difficult to see that $\|.\|_{H^p_\nu(\mathbb{D})}$ is a norm making $H^p_\nu(\mathbb{D})$ into a real Banach space.
When $\nu = 0$, then $H^p_\nu(\mathbb{D}) = H^p(\mathbb{D})$ viewed as a *real* vector space.

As we will see in Proposition 4.3.1, each $f \in H^p_\nu(\mathbb{D})$ has a non-tangential limit a e. on $\mathbb{T}$ that we call the trace of f, denoted by tr f (see Section 2 for the definition of the non-tangential limit). This definition causes no discrepancy since, as we shall see in Proposition 4.3.3 below, any solution of (5) in $W^{1,p}(\mathbb{D})$ belongs to $H^p_\nu(\mathbb{D})$ and, for an arbitrary function $f \in W^{1,p}(\mathbb{D})$, the nontangential limit of f, when it exists, coincides with the trace of f in the Sobolev sense. It turns out that, for all $f \in H^p_\nu(\mathbb{D})$, tr f lies in $L^p(\mathbb{T})$ and $\|\text{tr } f\|_{L^p(\mathbb{T})}$ defines an equivalent norm on $H^p_\nu(\mathbb{D})$.

We single out the subspace $H^{p,0}_\nu$ of H^p_ν consisting of those f for which

$$\int_0^{2\pi} \operatorname{Im}\left(\text{tr } f(e^{i\theta})\right) d\theta = 0 \tag{12}$$

holds. We further let $H^{p,00}_\nu$ be the subspace of functions $f \in H^{p,0}_\nu$ such that

$$\int_0^{2\pi} \text{tr } f(e^{i\theta})\, d\theta = 0\,. \tag{13}$$

Remark 3.2.1. *In what follows, we make use of both $H^p_\nu(\mathbb{D})$ and $H^p(\mathbb{D})$. For simplicity, we drop the dependence on $\mathbb{D}$ and denote them by H^p_ν and H^p, respectively. In particular, H^p (no subscript) always stands for the classical holomorphic Hardy space of the disk.*

3.2.2 The class $G^p_\alpha(\mathbb{D})$

For $\alpha \in L^\infty(\mathbb{D})$ and $1 < p < \infty$ (note that α may be complex-valued here), we define another space $G^p_\alpha(\mathbb{D}) = G^p_\alpha$, consisting of those Lebesgue measurable functions w on $\mathbb{D}$ such that:

$$\|w\|_{G^p_\alpha} := \operatorname*{ess\,sup}_{0<r<1} \|w\|_{L^p(\mathbb{T}_r)} < +\infty$$

and

$$\overline{\partial} w = \alpha \overline{w} \tag{14}$$

in the sense of distributions on $\mathbb{D}$. Note that $\|.\|_{H^p_\nu(\mathbb{D})}$ and $\|.\|_{G^p_\alpha(\mathbb{D})}$ formally coincide, but the equations (5) and (14) are different. Again $\|.\|_{G^p_\alpha(\mathbb{D})}$ makes G^p_α into a real Banach space. The reason why we introduce G^p_α is the tight connection it has with H^p_ν when we set $\alpha := -\bar{\partial}\nu/(1-\nu^2)$, as shown in Proposition 3.2.3.1 below. From equation (16) below, we see that α has this form for some $\nu \in W^{1,\infty}_{\mathbb{R}}$ meeting (6) if, and only if $\alpha = \bar{\partial} h$ for some $h \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$. Making such an assumption in the definition would be artificial, since most of the properties of G^p_α to come are valid as soon as $\alpha \in L^\infty(\mathbb{D})$. However, if (16) holds *and only in this case* (see section 3.2.3 below), we shall find it convenient to introduce the space $G^{p,0}_\alpha$ of those $w \in G^p_\alpha$ normalized by

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\sigma^{1/2}\mathrm{Im}\,\mathrm{tr}\ w\right)(e^{i\theta})\,d\theta = 0. \tag{15}$$

3.2.3 The link between H^p_ν and G^p_α

The explicit connection between H^p_ν and G^p_α is given by the following result, which relies on a transformation introduced in [20]:

Proposition 3.2.3.1. *Let $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ satisfy* (6) *and define $\sigma \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$, $\alpha \in L^\infty(\mathbb{D})$ by*

$$\sigma = \frac{1-\nu}{1+\nu}, \qquad \alpha := -\frac{\bar{\partial}\nu}{1-\nu^2} = \frac{\bar{\partial}\sigma}{2\sigma} = \bar{\partial}\log\sigma^{1/2}. \tag{16}$$

Then $f \in L^p(\mathbb{D})$ solves (5) *in the distributional sense if, and only if the function w, defined by*

$$w := (f - \nu\overline{f})/\sqrt{1-\nu^2} = \sigma^{1/2}\,u + i\,\sigma^{-1/2}\,v \tag{17}$$

does for (14)*. Moreover,*

(a) *$f = u + i\,v$ lies in H^p_ν (resp. $H^{p,0}_\nu$) if, and only if the function w given by (17) lies in G^p_α (resp. $G^{p,0}_\alpha$).*

(b) *$f \in W^{1,p}(\mathbb{D})$ solves* (5) *if, and only if w given by* (17) *solves* (14) *in $W^{1,p}(\mathbb{D})$.*

The proof is a straightforward computation, using that the distributional derivatives of $(f - \nu\overline{f})/\sqrt{1-\nu^2}$ can be computed by Leibniz's rule under our standing assumptions, and the fact that (17) can also be rewritten as

$$f = \frac{w + \nu\overline{w}}{\sqrt{1-\nu^2}}. \tag{18}$$

Observe that every constant $c \in \mathbb{C}$ is a solution to (5), the associated w *via* (17) being $\sigma^{1/2}\,\mathrm{Re}\ c + i\,\sigma^{-1/2}\,\mathrm{Im}\ c$, which lies in $W^{1,\infty}(\mathbb{D})$ and solves (14).

4 Statement of the results.

Throughout, we assume $1 < p < \infty$, and we let $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ satisfy $\|\nu\|_{L^\infty(\mathbb{D})} \leq \kappa < 1$.

4.1 Solvability in Sobolev spaces

Our first result deals with the solvability of the Dirichlet problem for (5) with boundary data in $W^{1-1/p,p}(\mathbb{T})$:

Theorem 4.1.1. *Let $p \in (1, +\infty)$ and $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ satisfy* (6)*.*

(*a*) *To each $\varphi \in W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$, there is $f \in W^{1,p}(\mathbb{D})$ solving* (5) *in $\mathbb{D}$ and such that* $\mathrm{Re}\,(tr\ f) = \varphi$ *on $\mathbb{T}$. Such an f is unique up to an additive pure imaginary constant.*

(*b*) *There exists $C_{p,\nu} > 0$ such that the function f in (a), when normalized by* (12)*, satisfies*

$$\|f\|_{W^{1,p}(\mathbb{D})} \leq C_{p,\nu} \|\varphi\|_{W^{1-1/p,p}(\mathbb{T})} . \tag{19}$$

Remark 4.1.1. *Although we will not use it, let us point out that Theorem 4.1.1 still holds if we merely assume $\nu \in VMO(\mathbb{D})$, provided* (5) *is understood in the pointwise sense. The proof is similar, appealing to [10] rather than [24] to solve the Dirichlet problem for* (4)*.*

Remark 4.1.2. *When $\varphi \in W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$ and $u \in W^{1,p}_{\mathbb{R}}(\mathbb{D})$ is the solution to* (9) *such that $tr\ u = \varphi$ granted by [24], the normal derivative $\partial_n u$ is classically* defined *as the unique member of the dual space $W^{-1/p,p}_{\mathbb{R}}(\mathbb{T}) = \left(W^{1-1/q,q}_{\mathbb{R}}(\mathbb{T})\right)^*$ such that*

$$\langle \partial_n u \,,\, \sigma\psi \rangle \;=\; \int_{\mathbb{D}} \sigma \nabla u . \nabla g \, dm \,, \qquad \psi \in W^{1-1/q,q}_{\mathbb{R}}(\mathbb{T}),\ g \in W^{1,q}(\mathbb{D}),\ tr\ g = \psi, \tag{20}$$

where g is any representative of the coset $tr^{-1}\psi$ in $W^{1,p}_{\mathbb{R}}(\mathbb{D})/W^{1,p}_{0,\mathbb{R}}(\mathbb{D})$. That $\partial_n u$ is well-defined via (20) *depends on the fact that M_σ, the multiplication by σ, is an isomorphism of $W^{1-1/q,q}_{\mathbb{R}}(\mathbb{T})$; this follows by interpolation since M_σ is an isomorphism both of $L^q_{\mathbb{R}}(\mathbb{T})$ and $W^{1,q}_{\mathbb{R}}(\mathbb{T})$. Now, if $f = u + iv \in W^{1,p}(\mathbb{D})$ is a solution to* (5) *such that* $\mathrm{Re}\,(tr\ f) = \varphi$ *as provided by Theorem 4.1.1, it is a straightforward consequence of* (8) *that $\partial_n u = (\partial_\theta tr\ v)/\sigma$.*

The results below generalize to H^p_ν and G^p_α, defined in Section 3.2, some fundamental properties of holomorphic Hardy classes [32, 35]. Observe that, on $\mathbb{D}$, as the above definition shows (see Section 2), f has a non tangential ("n.t.") limit ℓ at $e^{i\theta} \in \mathbb{T}$ if, and only if for every $0 < \beta < \pi/2$, $f(z)$ tends to ℓ as $z \to e^{i\theta}$ inside any sector $\Gamma_{e^{i\theta},\beta}$ with vertex $e^{i\theta}$, of angle 2β, which is symmetric with respect to the ray $(0, e^{i\theta})$. The non-tangential maximal function of f at $\xi \in \mathbb{T}$ is

$$\mathcal{M}_f(\xi) := \sup_{z \in \mathbb{D} \cap \Gamma_{\xi,\beta}} |f(z)|, \tag{21}$$

where we dropped the dependence of $\mathcal{M}_f$ on β.
We first mention properties of the class G^p_α, from which those of the class H^p_ν will be deduced using Proposition 3.2.3.1.

4.2 Properties of G^p_α

We fix $\alpha \in L^\infty(\mathbb{D})$. To proceed with the statements, we need to introduce two operators that will be of constant use in the paper. First, for $\psi \in L^1(\mathbb{T})$, we define a holomorphic function in $\mathbb{D}$ through the Cauchy operator:

$$\mathcal{C}\psi(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\psi(\xi)}{\xi - z} d\xi\,, \ z \in \mathbb{D}\,.$$

It follows from a theorem of M. Riesz that $\mathcal{C}$ maps $L^p(\mathbb{T})$ onto H^p, see the discussion after [35, Ch. 3, Thm 1.5]; this would fail if $p = 1, \infty$.
Second, for $p \in (1, +\infty)$ and $w \in L^p(\mathbb{D})$, we define

$$Tw(z) = \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{w(\xi)}{\xi - z} d\xi \wedge d\overline{\xi}\,, \ z \in \mathbb{D}\,.$$

The following representation theorem for functions in G^p_α was implicit in [20] for continuous $W^{1,2}(\mathbb{D})$-solutions to (14):

Theorem 4.2.1. *Let $w \in L^p(\mathbb{D})$ be a distributional solution to* (14). *Then w can be represented as*

$$w(z) = \exp(s(z))\, F(z)\,, \ z \in \mathbb{D}\,, \tag{22}$$

where $s \in W^{1,l}(\mathbb{D})$ for all $l \in (1, +\infty)$ and F is holomorphic in $\mathbb{D}$. Moreover, s can be chosen such that its real part (or else its imaginary part) is 0 on $\mathbb{T}$ and

$$\|s\|_{L^\infty(\mathbb{D})} \le 4\|\alpha\|_{L^\infty(\mathbb{D})}. \tag{23}$$

In particular $w \in W^{1,l}_{loc}(\mathbb{D})$ for all $l \in (1, +\infty)$, and $w \in G^p_\alpha$ if, and only if $F \in H^p$ in some, hence any factorization of the form (22). *Moreover, $w \in L^{p_1}(\mathbb{D})$, for all $p_1 \in [p, 2p)$.*

Remark 4.2.1. *By the Sobolev imbedding theorem ([2, Thm 5.4, Part II]), $s \in C^{0,\gamma}(\overline{\mathbb{D}})$ and $w \in C^{0,\gamma}_{loc}(\mathbb{D})$ for all $\gamma \in (0, 1)$.*

Theorem 4.2.1 will allow for us to carry over to G^p_α the essentials of the boundary behaviour of holomorphic Hardy functions:

Proposition 4.2.1. *1. If $w \in G^p_\alpha$, then*

$$tr\ w(e^{i\theta}) := \lim_{\xi \in \mathbb{D},\ \xi \to e^{i\theta}\ n.t.} w(\xi) \tag{24}$$

exists for almost every θ and

$$\|tr\ w\|_{L^p(\mathbb{T})} \le \|w\|_{G^p_\alpha} \le c_\alpha\, \|tr\ w\|_{L^p(\mathbb{T})} \tag{25}$$

for some $c_\alpha > 0$. Moreover,

$$\lim_{r \to 1} \int_0^{2\pi} \left|w(re^{i\theta}) - tr\ w(e^{i\theta})\right|^p d\theta = 0 \tag{26}$$

and, for any aperture $\beta \in (0, \pi/2)$ of the sectors used in definition (21), *there is a constant $C_{p,\alpha,\beta}$ such that*

$$\|\mathcal{M}_w\|_{L^p(\mathbb{T})} \le C_{p,\alpha,\beta} \|tr\ w\|_{L^p(\mathbb{T})}. \tag{27}$$

2. *If $w \in G^p_\alpha$ and $w \not\equiv 0$, then $\log |tr\ w| \in L^1(\mathbb{T})$; moreover the zeros of w are isolated in $\mathbb{D}$, and numbering them as $\alpha_1, \alpha_2, \cdots$, counting repeated multiplicities, it holds that*

$$\sum_{j=1}^{\infty}(1-|\alpha_j|) < +\infty. \tag{28}$$

3. *Let $w \in L^p(\mathbb{D})$. Then $w \in G^p_\alpha$ if, and only if there is a function $\varphi \in L^p(\mathbb{T})$ such that*

$$w = \mathcal{C}\varphi + T(\alpha\overline{w})\,,\ a.e.\ in\ \mathbb{D}\,. \tag{29}$$

In this situation,

$$\|w\|_{G^p_\alpha} \leq C_{p,\alpha}\left(\|w\|_{L^p(\mathbb{D})} + \|\varphi\|_{L^p(\mathbb{T})}\right). \tag{30}$$

A valid choice in (29) *is $\varphi = tr\ w$.*

4. *If $w \in G^p_\alpha$ satisfies* (12) *and Re $tr\ w = 0$ a.e. on $\mathbb{T}$, then $w \equiv 0$ in $\mathbb{D}$. When* (16) *holds, the same is true if $w \in G^{p,0}_\alpha$.*

Remark 4.2.2. *From* (25) *and the completeness of G^p_α, we see that tr G^p_α is a closed subspace of $L^p(\mathbb{T})$. We also observe, in view of the M. Riesz theorem, that assertion 3 can be recaped as: $w \in G^p_\alpha \Longleftrightarrow w - T(\alpha\overline{w}) \in H^p$.*

Theorem 4.2.1 and Proposition 4.2.1 are proven in Section 5.3.

4.3 Properties of the Hardy class H^p_ν

Proposition 4.3.1. *The following statements hold true.*

(a) *If $f \in H^p_\nu$, then f has a non-tangential limit a.e. on $\mathbb{T}$, denoted by $tr\ f$, the $L^p(\mathbb{T})$-norm of which is equivalent to the H^p_ν-norm of f:*

$$\|tr\ \ f\|_{L^p(\mathbb{T})} \leq \|f\|_{H^p_\nu(\mathbb{D})} \leq c_\nu\ \|tr\ \ f\|_{L^p(\mathbb{T})}\ . \tag{31}$$

Moreover

$$\lim_{r\to 1}\int_0^{2\pi}\left|f(re^{i\theta}) - tr\ f(e^{i\theta})\right|^p d\theta = 0,$$

and we have that $f \in L^{p_1}(\mathbb{D})$ for $p \leq p_1 < 2p$.

(b) *The image space tr H^p_ν (resp. tr $H^{p,0}_\nu$) is closed in $L^p(\mathbb{T})$.*

(c) *Each $f \in H^p_\nu$ is such that $\log|tr\ \ f| \in L^1(\mathbb{T})$ unless $f \equiv 0$.*

(d) *If $f \in H^p_\nu$ and $f \not\equiv 0$, then its zeros are isolated in $\mathbb{D}$; if we enumerate them as $\alpha_1, \alpha_2, \cdots$, counting repeated multiplicities, then* (28) *holds.*

(e) *For any aperture $\beta \in (0, \pi/2)$ of the sectors used in definition* (21)*, there is a constant $C_{p,\nu,\beta}$ such that*

$$\|\mathcal{M}_f\|_{L^p(\mathbb{T})} \leq C_{p,\nu,\beta}\|tr\ \ f\|_{L^p(\mathbb{T})}.$$

(f) Each $f \in H^p_\nu$ satisfies the maximum principle, i.e. $|f|$ cannot assume a relative maximum in $\mathbb{D}$ unless it is constant. More generally, a non constant function in H^p_ν is open and discrete[3].

It is rather easy to deduce Proposition 4.3.1 from the corresponding properties for the $G^p_\alpha(\mathbb{D})$ class. Indeed, by (6) and Proposition 3.2.3.1 we can invert (17) by (18), so that items (a)-(e) follow at once from Proposition 4.2.1, Remark 4.2.2, and the fact that f and w share the same zeros because

$$w = \frac{f - \nu\overline{f}}{\sqrt{1-\nu^2}}.$$

To prove (f), observe from Theorem 4.2.1 that w, thus also f belongs to $W^{1,l}_{loc}(\mathbb{D})$ for each $1 < l < \infty$. Moreover if we let $\nu_f(z) := \nu(z)\overline{\partial f(z)}/\partial f(z)$if $\partial f(z) \neq 0$ and $\nu_f(z) = 0$ otherwise, then f is a pointwise a.e. solution in $\mathbb{D}$ of the *classical* Beltrami equation:

$$\overline{\partial} f = \nu_f \, \partial f, \qquad |\nu_f| \leq \kappa < 1. \tag{32}$$

Indeed, $|\nu(z)| \leq \kappa$ for all $z \in \mathbb{D}$ and $\left|\overline{\partial f(z)}/\partial f(z)\right| = 1$ when $\partial f(z) \neq 0$. It is then a standard result [43, Thm 11.1.2] that $f = G(h(z))$, where h is a quasi-conformal topological map $\mathbb{D} \to \mathbb{C}$ satisfying (32) and G a holomorphic function on $h(\mathbb{D})$. The conclusion follows at once from the corresponding properties of holomorphic functions. □

Remark 4.3.1. *When $\nu = 0$, that is, when dealing with holomorphic Hardy spaces, the best constant in* (31) *is $c_0 = 1$ because $\|f\|_{L^p(\mathbb{T}_r)}$ increases with r, and then equality holds throughout. For general ν, a bound on c_ν depending solely on $\|\nu\|_{W^{1,\infty}(\mathbb{D})}$ is easily derived from Proposition 3.2.3.1 and Theorem 4.2.1, but the authors do not know of a sharp estimate.*

Remark 4.3.2. *Assertion (c) in Proposition 4.3.1 implies that a function $f \in H^p_\nu(\mathbb{D})$ whose trace is zero on a subset of $\mathbb{T}$ having positive Lebesgue measure must vanish identically.*

As in the holomorphic case, a function in $H^{p,0}_\nu$ is uniquely defined by its real part on $\mathbb{T}$:

Proposition 4.3.2. *Let $f \in H^{p,0}_\nu$ be such that Re $(tr\ f) = 0$ a.e. on $\mathbb{T}$. Then $f \equiv 0$.*

Proof of Proposition 4.3.2. Let $f \in H^{p,0}_\nu(\mathbb{D})$ satisfy $\mathrm{Re}\,\mathrm{tr}\ f = 0$ a.e. on $\mathbb{T}$. If we define w through (17), then $w \in G^{p,0}_\alpha$ by Proposition 3.2.3.1 and clearly $\mathrm{Re}\,\mathrm{tr}\ w = 0$ a.e. on $\mathbb{T}$. Therefore $w \equiv 0$ in view of Proposition 4.2.1, assertion 4, whence $f \equiv 0$ in $\mathbb{D}$. □

The next result shows that H^p_ν contains all $W^{1,p}(\mathbb{D})$ solutions to (5). That this inclusion is a strict one follows at once from Theorem 4.4.2.1 to come.

Proposition 4.3.3. *Let $f \in W^{1,p}(\mathbb{D})$ be a solution to (5). Then $f \in H^p_\nu(\mathbb{D})$, and there exists $C_{\nu,p} > 0$ such that,*

$$\|f\|_{H^p_\nu(\mathbb{D})} \leq C_{\nu,p} \, \|f\|_{W^{1,p}(\mathbb{D})} \,. \tag{33}$$

Moreover, the trace of f considered as an element of $W^{1,p}(\mathbb{D})$ coincides with its trace seen as an element of $H^p_\nu(\mathbb{D})$.

Note that (33) follows immediately from (31) and the continuity of the trace operator from $W^{1,p}(\mathbb{D})$ into $L^p(\mathbb{T})$, once it is known that $f \in H^p_\nu$.

The proof of Proposition 4.3.3 is given in Section 5.4.

[3] A map is discrete if the preimage of any value is a discrete subset of its domain.

4.4 Regularity of the Dirichlet problem

4.4.1 Solvability of the Dirichlet problem in $G^p_\alpha(\mathbb{D})$

We first focus on a slight variation of the Dirichlet problem for the $G^p_\alpha(\mathbb{D})$ class. Let us introduce one more piece of notation by letting

$$P_+\psi = \text{tr}\ \ (\mathcal{C}\psi)$$

where the trace is a nontangential limit. As is well-known, $P_+\psi$ exists a.e. on $\mathbb{T}$ as soon as $\psi \in L^1(\mathbb{T})$, but it may not lie in $L^1(\mathbb{T})$. If, however, $1 < p < \infty$, then P_+ is a continuous projection from $L^p(\mathbb{T})$ onto tr H^p called the analytic projection [35, Ch. III, Sec. 1]. It is an interesting variant of the Dirichlet problem to solve equation (14) while prescribing the analytic projection of the solution on $\mathbb{T}$. As the next theorem shows, G^p_α is a natural space for this.

Theorem 4.4.1.1. *For $\alpha \in L^\infty(\mathbb{D})$ and $g \in H^p$, there is a unique $w \in G^p_\alpha$ such that*

$$P_+(tr\ w) = tr\ \ g\,. \tag{34}$$

This solution satisfies

$$w = g + T(\alpha\overline{w})\,,\ \ a.e.\ \ in\ \mathbb{D}\,, \tag{35}$$

and it holds that

$$\|w\|_{G^p_\alpha} \leq C_{p,\alpha}\,\|g\|_{H^p(\mathbb{D})}\,. \tag{36}$$

Here is now the solution of the (usual) Dirichlet problem for the class G^p_α:

Theorem 4.4.1.2. *Let $\alpha \in L^\infty(\mathbb{D})$ and $\psi \in L^p_\mathbb{R}(\mathbb{T})$.*

(a) To each $c \in \mathbb{R}$, there uniquely exists $w \in G^p_\alpha$ such that $Re\ (tr\ \ w) = \psi$ a.e. on $\mathbb{T}$ and $\int_0^{2\pi} Im\,tr\ \ w(e^{i\theta})\,d\theta = 2\pi c$. Moreover there are constants $c_{p,\alpha}$ and $c'_{p,\alpha}$ such that

$$\|tr\ \ w\|_{L^p(\mathbb{T})} \leq c_{p,\alpha}\,\|\psi\|_{L^p(\mathbb{T})} + c'_{p,\alpha}\,|c|\,. \tag{37}$$

(b) When (16) holds, there uniquely exists $w \in G^{p,0}_\alpha$ such that $Re\ (tr\ \ w) = \psi$ a.e. on $\mathbb{T}$. Furthermore, there is a constant $c''_{p,\alpha}$ such that

$$\|tr\ \ w\|_{L^p(\mathbb{T})} \leq c''_{p,\alpha}\,\|\psi\|_{L^p(\mathbb{T})}\ . \tag{38}$$

The proofs of Theorems 4.4.1.1-4.4.1.2 are given in section 5.5.

4.4.2 Solvability of the Dirichlet problem in $H^p_\nu(\mathbb{D})$

The following result shows that H^p_ν is the natural space to consider when handling L^p boundary data in (5) and (4).

Theorem 4.4.2.1. *For all $\varphi \in L^p_\mathbb{R}(\mathbb{T})$, there uniquely exists $f \in H^{p,0}_\nu$ such that, a.e. on $\mathbb{T}$:*

$$Re\ (tr\ \ f) = \varphi\,. \tag{39}$$

Moreover, there exists $c_{p,\nu} > 0$ such that:

$$\|f\|_{H^p_\nu(\mathbb{D})}\ \leq c_{p,\nu}\,\|\varphi\|_{L^p(\mathbb{T})}\ . \tag{40}$$

From Proposition 4.3.3, Theorem 4.4.2.1 clearly extends Theorem 4.1.1 when the boundary data belong to $L^p(\mathbb{T})$.

Let us give at once the proof of Theorem 4.4.2.1, which is quite easy to deduce from previous statements. Define α through (16) and put $\psi = \varphi\sigma^{1/2} \in L^p_{\mathbb{R}}(\mathbb{T})$. Apply Theorem 4.4.1.2, point (b), to obtain $w \in G^{p,0}_\alpha$ such that Re (tr w) $= \psi$. If we let f be given by (18), then $f \in H^{p,0}_\nu$ by Proposition 3.2.3.1, point (a). Moreover, from (17), we see that (39) holds. The uniqueness of f comes from Proposition 4.3.2. Inequality (40) follows from (18), (3), (38) and (25). □

Dwelling on Proposition 4.3.1 and Theorem 4.4.2.1, we are now able to derive an analog of the Fatou theory [35, Ch. I, Sec. 5] for (4), at least when $1 < p < \infty$. It should be compared to classical results on the Dirichlet problem in Sobolev classes [24, 36]. For once, we recall all the assumptions to ease this comparison.

Theorem 4.4.2.2. *Let $1 < p < \infty$ and $\sigma \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ satisfy (3). Any $u \in L^p_{\mathbb{R}}(\mathbb{D})$ satisfying (4) in the sense of distributions lies in $W^{1,l}_{\mathbb{R},loc}(\mathbb{D})$ for all $l \in (1,\infty)$. If, moreover,*

$$\|u\|_{F,p} := \underset{0<r<1}{ess\ sup}\ \|u\|_{L^p(\mathbb{T}_r)} < +\infty\,, \tag{41}$$

then u has a nontangential limit $tr\ u$ a.e. on $\mathbb{T}$ which is also the limit of $e^{i\theta} \mapsto u(re^{i\theta})$ in $L^p_{\mathbb{R}}(\mathbb{T})$ as $r \to 1^-$. In this case, for $\mathcal{M}_{|u|}$ the non tangential maximal function, we have

$$\|tr\ u\|_{L^p_{\mathbb{R}}(\mathbb{T})} \le \|u\|_{F,p} \le c_{p,\nu}\|\mathcal{M}_{|u|}\|_{L^p(\mathbb{T})} \le C_{p,\nu}\|tr\ u\|_{L^p_{\mathbb{R}}(\mathbb{T})}.$$

Conversely, each member of $L^p_{\mathbb{R}}(\mathbb{T})$ is uniquely the non tangential limit of some distributional solution $u \in L^p(\mathbb{D})$ of (4) satisfying $\|u\|_{F,p} < +\infty$.

The proof of Theorem 4.4.2.2 is carried out in Section 5.6.
Theorem 4.4.2.1 allows one to define a generalized conjugation operator $\mathcal{H}_\nu$ from $L^p(\mathbb{T})$ into itself, that was introduced on $W^{1/2,2}(\mathbb{T})$ in [8] as the ν-Hilbert transform. More precisely, to each $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$, we associate the unique function $f \in H^{p,0}_\nu$ such that Re tr $f = \varphi$, and we set $\mathcal{H}_\nu\varphi = \mathrm{Im}\, tr f \in L^p(\mathbb{T})$. It now follows from Theorems 4.1.1 and 4.4.2.1 that:

Corollary 4.4.2.1. *The operator $\mathcal{H}_\nu$ is bounded both on $L^p_{\mathbb{R}}(\mathbb{T})$ and on $W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$.*

When $\nu = 0$, we observe that $\mathcal{H}_0\,\varphi$ is just the harmonic conjugate[4] of φ normalized to have zero mean on $\mathbb{T}$. That the operator $\mathcal{H}_0$ is continuous from $L^p_{\mathbb{R}}(\mathbb{T})$ into itself is the well-known M. Riesz theorem [35, Ch. III, thm. 2.3]. Corollary 4.4.2.1 thus generalizes the latter.

4.4.3 Improved regularity results for the Dirichlet problem

We turn to higher regularity for solutions to (5). More precisely, we shall study the improvement in the conclusion of Theorem 4.4.2.1 when the boundary condition lies in $W^{1,p}_{\mathbb{R}}(\mathbb{T}) \subset W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$. First, the generalized conjugation operator preserves this smoothness class (compare Corollary 4.4.2.1):

[4] Though it has the same behaviour, it is distinct from the Hilbert transform, see [35, Chap. III, So. 1].

Proposition 4.4.3.1. *The operator $\mathcal{H}_\nu$ is bounded on $W^{1,p}_{\mathbb{R}}(\mathbb{T})$.*

Next, assuming in Theorem 4.4.2.1 that $\varphi \in W^{1,p}_{\mathbb{R}}(\mathbb{T})$, not only does f belong to $W^{1,p}(\mathbb{D})$, as predicted by Theorem 4.1.1, but the derivatives of f satisfy a condition of Hardy type:

Theorem 4.4.3.1. *Let $\varphi \in W^{1,p}_{\mathbb{R}}(\mathbb{T})$ and $f \in W^{1,p}(\mathbb{D})$ be the unique solution to (5) on $\mathbb{D}$ satisfying $\operatorname{Re}(tr\ f) = \varphi$ and such that (12) holds. Then,*

(a) $tr\ f \in W^{1,p}(\mathbb{T})$, and it holds that

$$\|tr\ f\|_{W^{1,p}(\mathbb{T})} \leq C_{p,\nu} \|\varphi\|_{W^{1,p}(\mathbb{T})}. \tag{42}$$

(b) The functions ∂f and $\overline{\partial} f$ satisfy a Hardy condition of the form

$$\operatorname*{ess\ sup}_{0<r<1} \|\partial f\|_{L^p(\mathbb{T}_r)} \leq C_{p,\nu} \|tr\ f\|_{W^{1,p}(\mathbb{T})}, \tag{43}$$

$$\operatorname*{ess\ sup}_{0<r<1} \left\|\overline{\partial} f\right\|_{L^p(\mathbb{T}_r)} \leq C_{p,\nu} \|tr\ f\|_{W^{1,p}(\mathbb{T})}, \tag{44}$$

and for the non tangential maximal function of $\|\nabla f\|$, it holds that

$$\|\mathcal{M}_{\|\nabla f\|}\|_{L^p(\mathbb{T})} \leq C_{p,\nu,\beta} \|tr\ f\|_{W^{1,p}(\mathbb{T})}, \tag{45}$$

where $\nabla f(\xi) \in \mathbb{C}^2$ is the gradient of f and β the aperture of the sectors in (21).

(c) If we define $\Phi \in L^p(\mathbb{T})$ by

$$\Phi(e^{i\theta}) := -ie^{-i\theta} \, \frac{\partial_\theta (tr\ f)(e^{i\theta}) - \nu(e^{i\theta}) \partial_\theta \overline{(tr\ f)(e^{i\theta})}}{1 - \nu^2(e^{i\theta})}, \tag{46}$$

then ∂f and $\overline{\partial} f$ have non tangential limit Φ and $\nu\overline{\Phi}$ a.e. on $\mathbb{T}$, and $\partial f(re^{i\theta})$, $\overline{\partial} f(re^{i\theta})$ converge in $L^p(\mathbb{T})$ to their respective nontangential limits as $r \to 1$.

Clearly, (a) is a rephrasing of Proposition 4.4.3.1.

As a corollary of Theorem 4.4.3.1, we obtain the following result (compare [33]), which plays for the Neumann problem the same role as Theorem 4.4.2.2 does for the Dirichlet problem:

Corollary 4.4.3.1. 1. *Let $u \in W^{1,p}_{\mathbb{R}}(\mathbb{D})$ be a solution of div $(\sigma \nabla u) = 0$ in $\mathbb{D}$ such that ∇u satisfies the following Hardy condition:*

$$\operatorname*{ess\ sup}_{0<r<1} \|\nabla u\|_{L^p(\mathbb{T}_r)} < +\infty. \tag{47}$$

Then $tr\ u \in W^{1,p}(\mathbb{T})$ and $u \in W^{1,p_1}_{\mathbb{R}}(\mathbb{D})$ for every $p_1 \in (p, 2p)$. Moreover, $\mathcal{M}_{\|\nabla u\|} \in L^p(\mathbb{T})$, and there exists a vector field $\Phi \in L^p(\mathbb{T}, \mathbb{R}^n)$ such that $\nabla u \to \Phi$ n.t. almost everywhere on $\mathbb{T}$. In particular, $\partial_n u \in L^p(\mathbb{T})$, and one has $\int_{\mathbb{T}} \sigma \partial_n u = 0$.

2. *Conversely, if $g \in L^p_{\mathbb{R}}(\mathbb{T})$ satisfies $\int_{\mathbb{T}} \sigma g = 0$, there exists a function $u \in W^{1,p}_{\mathbb{R}}(\mathbb{D})$ solving div $(\sigma \nabla u) = 0$ in $\mathbb{D}$ such that ∇u satisfies a Hardy condition of the form (47), $\mathcal{M}_{\|\nabla u\|} \in L^p(\mathbb{T})$ and $\partial_n u = g$ on $\mathbb{T}$. Moreover, u is unique up to an additive constant.*

All these results will be established in Section 5.7.

4.5 Density of traces

We come to some density properties of traces of solutions to (5). Loosely speaking, they assert that if $E \subset \mathbb{T}$ is not too large, then every complex function on E can be approximated by the trace of a solution to (5) on $\mathbb{D}$.

4.5.1 Density in Sobolev spaces

We say that an open subset I of T has the *extension property* if every function in $W^{1,p}(I)$ is the restriction to I of some function in $W^{1,p}(\mathbb{T})$. If I is a proper open subset of $\mathbb{T}$, it decomposes into a countable union of disjoint open arcs (a_j, b_j) and the extension property is equivalent to the fact that no a_j (resp. b_j) is a limit point of the sequence (b_k) (resp. (a_k)).
We begin with a density property of Sobolev solutions to (5) on proper extension subsets:

Theorem 4.5.1.1. *Let $I \neq \mathbb{T}$ be an open subset of T having the extension property. Then, the restrictions to I of traces of $W^{1,p}(\mathbb{D})$-solutions to* (5) *form a dense subspace of $W^{1-1/p,p}(I)$.*

This should be held in contrast with the fact that the traces on $\mathbb{T}$ of $W^{1,p}(\mathbb{D})$-solutions to (5) form a proper closed subspace of $W^{1-1/p,p}(\mathbb{T})$.
The proof of Theorem 4.5.1.1 is given in Section 5.8.1.

4.5.2 Density in Lebesgue spaces

By the density of $W^{1-1/p,p}(I)$ in $L^p(I)$, Theorem 4.5.1.1 easily implies that $(\text{tr } H^p_\nu)_{|_I}$ is a dense subset of $L^p(I)$ for I a proper open subset of $\mathbb{T}$ having the extension property. The fact that this remains true as soon as I is not of full measure lies a little deeper:

Theorem 4.5.2.1. *Let $I \subset \mathbb{T}$ be a measurable subset such that $\mathbb{T} \setminus I$ has positive Lebesgue measure. The restrictions to I of traces of H^p_ν-functions are dense in $L^p(I)$.*

Remark 4.5.2.1. *When $I \subset \mathbb{T}$ is not of full measure and $\phi \in L^p(I)$, Theorem 4.5.2.1 entails there is a sequence of functions $f_k \in H^p_\nu$ whose trace on I converges to ϕ in $L^p(I)$. Now, since balls in tr H^p_ν are weakly compact by Proposition 4.3.1 point (b), it must be that either ϕ is the trace on I of a H^p_ν-function or $\|tr\ f_k\|_{L^p(\mathbb{T}\setminus I)} \to +\infty$ with k. In view of Theorem 4.5.1.1, the corresponding remark applies when I is an open subset of $\mathbb{T}$ with the extension property and $\varphi \in W^{1-1/p,p}(I)$ gets approximated in this space by a sequence of traces of $W^{1,p}(\mathbb{D})$-solutions to* (5).

It is worth recasting Remark 4.5.2.1 in terms of ill-posedness of the inverse Dirichlet-Neumann problem from incomplete boundary data. Indeed, assume that u satisfies (4) and, say, tr $u \in W^{1,p}(\mathbb{T})$. Observe that the normal derivative $\partial_n u$ exists as a nontangential limit in $L^p(\mathbb{T})$ by Theorem 4.4.3.1. Thus, upon rewriting (8) on $\mathbb{T}$ in the form

$$\begin{cases} \partial_n v = -\sigma \partial_\theta u, \\ \partial_\theta v = \sigma \partial_n u, \end{cases} \tag{48}$$

we see that the knowledge of tr u and tr $\partial_n u$ on some arc $I \subset \mathbb{T}$ is equivalent to the knowledge on I of tr f where $f = u + iv$ meets (5) with boundary conditions Re$f = u$

and, say, $\int_I v = 0$. Note from Proposition 4.3.1 that this determines f completely. Now, if the knowledge of tr f gets corrupted by measurements and rounding off errors, as is the case in computational and engineering practice, it can still be approximated arbitrarily well in $W^{1-1/p,p}(I)$ by a solution to (5) but the trace of the latter will grow large in $L^p(\mathbb{T} \setminus I)$, *a fortiori* in $W^{1-1/p,p}(\mathbb{T} \setminus I)$ when the approximation error gets small.
The proof of Theorem 4.5.2.1 is carried out in Section 5.8.2.

4.6 Duality

Keeping in mind that $1 < p < \infty$ and $1/p + 1/q = 1$, we introduce a duality pairing on $L^p(\mathbb{T}) \times L^q(\mathbb{T})$, viewed as real vector spaces, by the formula:

$$\langle f, g \rangle = \mathrm{Re} \int_0^{2\pi} f\, g \,\frac{d\theta}{2\pi}. \tag{49}$$

Clearly this pairing isometrically identifies $L^q(\mathbb{T})$ with the dual of $L^p(\mathbb{T})$. The fact that H^p is the orthogonal space to $e^{i\theta} H^q$ under (49) is basic to the dual approach of extremal problems in holomorphic Hardy spaces [32, Ch. 8]. In this section, we derive the corresponding results for the spaces H^p_ν. Recall that $\partial_t = \partial_\theta / 2\pi$ on $\mathbb{T}$.

Proposition 4.6.1. *The orthogonal to tr H^p_ν under the duality pairing defined in* (49) *is*

$$(tr\ \ H^p_\nu)^\perp = \partial_\theta \left(tr\ \ H^q_{-\nu} \cap W^{1,q}(\mathbb{T}) \right) \ .$$

Proposition 4.6.1 and the Hahn-Banach theorem now team up to yield:

Theorem 4.6.1.

(i) Under the pairing (49)*, the dual space $(tr\ \ H^p_\nu)^*$ of tr H^p_ν is naturally isometric to the quotient space $L^q(\mathbb{T})/(tr\ \ H^p_\nu)^\perp$, that is*

$$(tr\ \ H^p_\nu)^* \sim L^q(\mathbb{T}) / \left(\partial_\theta \left(tr\ \ H^q_{-\nu} \cap W^{1,q}(\mathbb{T}) \right) \right) \ .$$

(ii) For each $\Phi \in L^q(\mathbb{T})$, it holds the duality relation

$$\inf_{g \in \partial_\theta (tr\ \ H^q_{-\nu} \cap W^{1,q}(\mathbb{T}))} \|\Phi - g\|_{L^q(\mathbb{T})} = \sup_{\substack{f \in H^p_\nu \\ \|tr\ \ f\|_{L^p(\mathbb{T})=1}}} \frac{1}{2\pi} \mathrm{Re} \int_0^{2\pi} \Phi\, tr\ \ f\, d\theta. \tag{50}$$

(iii) For each $\Psi \in L^p(\mathbb{T})$, it holds the duality relation

$$\inf_{f \in H^p_\nu} \|\Psi - tr\ \ f\|_{L^p(\mathbb{T})} = \sup_{\substack{g \in tr\ \ H^q_{-\nu} \cap W^{1,q}(\mathbb{T}) \\ \|\partial_\theta g\|_{L^q(\mathbb{T})} = 1}} \frac{1}{2\pi} \mathrm{Re} \int_0^{2\pi} \Psi\, \partial_\theta g\, d\theta. \tag{51}$$

Granted Proposition 4.6.1, Theorem 4.6.1 is a standard application of the Hahn-Banach theorem [32, Thms 7.1, 7.2]. □

Remark 4.6.1. *It is easy to check (compare [32, Thm 3.11]) that $\partial_\theta(tr\ \ H^q \cap W^{1,q}(\mathbb{T})) = e^{i\theta} H^q$, hence* (50)-(51) *reduce to standard duality relations in Hardy spaces when $\nu = 0$.*

The proofs of Proposition 4.6.1 and Theorem 4.6.1 will be given in Section 5.9.

5 Proofs

5.1 The Dirichlet problem in Sobolev spaces

Let us give first the proof of Theorem 4.1.1. Put $\sigma = (1-\nu)/(1+\nu)$, so that $\sigma \in W^{1,\infty}(\mathbb{D})$ satisfies (3). By [24], there uniquely exists $u \in W^{1,p}(\mathbb{D})$ meeting tr $u = \varphi$ for which (9) holds with $\Omega = \mathbb{D}$; moreover, by the open mapping theorem, one has

$$\|u\|_{W^{1,p}(\mathbb{D})} \leq c_{p,\nu} \, \|\varphi\|_{W^{1-1/p,p}(\mathbb{T})} \, . \tag{52}$$

Put

$$\mathcal{G}_{0,p} := \{\nabla g;\ g \in W^{1,p}_{0,\mathbb{R}}(\mathbb{D})\} \quad \text{and} \quad \mathcal{D}_q := \{(\partial_y h, -\partial_x h)^T;\ h \in W^{1,q}_{\mathbb{R}}(\mathbb{D})\}. \tag{53}$$

Proceeding by density on the divergence formula for smooth functions, we easily get

$$\langle G, D\rangle := \int_{\mathbb{D}} G.D \, dm = 0, \qquad G \in \mathcal{G}_{0,p},\ D \in \mathcal{D}_q. \tag{54}$$

Now, by Hodge theory [43, Thm 10.5.1][5], each vector field in $L^p(\mathbb{D}) \times L^p(\mathbb{D})$ (resp. $L^q(\mathbb{D}) \times L^q(\mathbb{D})$) is uniquely the sum of a member of $\mathcal{G}_{0,p}$ (resp. $\mathcal{G}_{0,q}$) and a member of $\mathcal{D}_p$ (resp. $\mathcal{D}_q$). If we set accordingly $\sigma\nabla u = G + D$, we gather by density from (9) and (54) that $\langle G, V\rangle = 0$ for every $V \in L^q(\mathbb{D}) \times L^q(\mathbb{D})$, implying that $\sigma\nabla u \in \mathcal{D}_p$. In other words there is $v \in W^{1,p}_{\mathbb{R}}(\mathbb{D})$ for which (8) holds, thus by inspection $f = u + iv \in W^{1,p}(\mathbb{D})$ satisfies (5) pointwise a.e. on $\mathbb{D}$. Then, f satisfies (5) in the distributional sense as well. To check f is unique, subject to $\mathrm{Re}\,\mathrm{tr}\ f = \varphi$, up to an additive pure imaginary constant, observe if $f \in W^{1,p}(\mathbb{D})$ satifies (5) that $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$ both lie in $W^{1,p}_{\mathbb{R}}(\mathbb{D})$ and that (8) holds. Therefore $\sigma\nabla u \in \mathcal{D}_p$ and, in view of (54), we see that (9) holds with $\Omega = \mathbb{D}$. As such a u is uniquely defined by tr $u = \varphi$, we conclude that v is uniquely defined by (8), up to an additive constant.
Next,we observe from (8) and (3) that $\|\nabla v\|_{L^p(\mathbb{D})} \leq C\|\nabla u\|_{L^p(\mathbb{D})}$. Therefore, if v gets normalized by (12), it follows from (52) and the Poincaré inequality [67, Ch. 4, Ex. 4.10] that

$$\|v\|_{W^{1,p}(\mathbb{D})} \leq C_p \|\nabla v\|_{L^p(\mathbb{D})} \leq C C_p c_{p,\nu} \, \|\varphi\|_{W^{1-1/p,p}(\mathbb{T})}$$

so that (19) indeed holds. □

5.2 Preliminaries on spaces and operators

In the present subsection, we recall some properties of the operators $\mathcal{C}$ and T, introduced in Section 4.2, and of the Beurling operator appearing in equation (55) below.
For $h \in L^p(\mathbb{C})$, we define the operator $\check{T}$ by

$$\check{T}h(z) = \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{h(\xi)}{\xi - z} d\xi \wedge d\overline{\xi} \, , \quad z \in \mathbb{C} \, .$$

If $w \in L^p(\mathbb{D})$ and $\breve{w}$ is the extension of w by 0 outside $\mathbb{D}$, then obviously $(\check{T}\breve{w})_{|\mathbb{D}} = Tw$.
Next, for $u \in L^p(\mathbb{C})$, we denote by S the Beurling operator:

$$Su(z) = \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \iint_{\xi \in \mathbb{C},\ |\xi - z| \geq \varepsilon} \frac{u(\xi)}{(\xi - z)^2} d\xi \wedge d\overline{\xi} \, , \quad \text{a.e. } z \in \mathbb{C} \, . \tag{55}$$

[5] The result is stated there using the language of differential forms that we did not introduce here.

The existence of Su a.e. follows from the Calderòn-Zygmund theory of singular integral operators [64, Ch. II, Thm 4]. Here are the properties of $\mathcal{C}$, $\breve{T}$, T and S that we use:

Proposition 5.2.1. *Let as usual $1 < p < +\infty$. Then the following assertions hold.*

1. *The Cauchy operator $\mathcal{C}$ is bounded from $L^p(\mathbb{T})$ onto $H^p(\mathbb{D})$ and from $W^{1-1/p,p}(\mathbb{T})$ to $W^{1,p}(\mathbb{D})$.*

2. *The Beurling operator S is bounded from $L^p(\mathbb{C})$ into itself.*

3. *The operator $\breve{T}$ maps $L^p(\mathbb{C})$ continuously into $W^{1,p}_{loc}(\mathbb{C})$.*

4. *The operator T is bounded from $L^p(\mathbb{D})$ into $W^{1,p}(\mathbb{D})$, and is compact from $L^p(\mathbb{D})$ to $L^p(\mathbb{D})$. Moreover $\bar{\partial}Tw = w$ and $\partial Tw = (S\breve{w})_{|\mathbb{D}}$ for all $w \in L^p(\mathbb{D})$. For any $\alpha \in L^\infty(\mathbb{D})$ the operator $w \mapsto w - T(\alpha\overline{w})$ is an isomorphism of $L^p(\mathbb{D})$.*

The next result will be of technical importance to establish the regularity properties of G^p_α-functions, compare Remark 4.2.2.

Lemma 5.2.1. *Let $p \in (1, +\infty)$ as always, and $\alpha \in L^\infty(\mathbb{D})$.*

1. *If $g \in H^p(\mathbb{D})$, then $g \in L^{p_1}(\mathbb{D})$ for $p_1 \in [p, 2p)$.*

2. *If $w \in L^p(\mathbb{D})$ and if $w - T(\alpha\overline{w}) \in H^p$, then there is $p^* > 2$ such that $T(\alpha\overline{w}) \in W^{1,p^*}(\mathbb{D}) \subset C^{0,1-2/p^*}(\overline{\mathbb{D}})$ and $\breve{T}\left(\breve{\alpha\overline{w}}\right) \in W^{1,p^*}_{loc}(\mathbb{C}) \subset C^{0,1-2/p^*}_{loc}(\mathbb{C})$. Moreover,*

$$\|T(\alpha\overline{w})\|_{W^{1,p^*}(\mathbb{D})} \leq C_{p,\alpha}\left(\|w\|_{L^p(\mathbb{D})} + \|w - T(\alpha\overline{w})\|_{H^p(\mathbb{D})}\right). \tag{56}$$

In order not to disrupt the reading, we postpone the proofs of Proposition 5.2.1 and Lemma 5.2.1 to Appendix A.

5.3 Factorization and boundary behaviour in G^p_α

This section is devoted to the proof of Theorem 4.2.1 and Proposition 4.2.1. In the proof of the former, we make use of the following Lemma.

Lemma 5.3.1. *Let $r \in L^\infty(\mathbb{C})$ be supported in $\mathbb{D}$. Then, the function*

$$u(z) = \iint_{\mathbb{D}} \frac{z\overline{r(\zeta)}}{1 - \bar{\zeta}z} d\zeta \wedge d\bar{\zeta}, \quad z \in \mathbb{C},$$

is holomorphic in $\mathbb{D}$ and belongs to $W^{1,l}_{loc}(\mathbb{C})$ for all $l \in (1, +\infty)$.

Proof of Lemma 5.3.1. The function u is clearly holomorphic in $\mathbb{D}$, *a fortiori* $u \in W^{1,l}_{loc}(\mathbb{D})$. It is therefore enough to show, say, that $u \in W^{1,l}_{loc}(\mathbb{C} \setminus \overline{\mathbb{D}}_{1/2})$ for all $l \in (1, +\infty)$. In turn, it is sufficient to prove that

$$u(1/\overline{z}) = -\overline{\int_{\mathbb{D}} \frac{r(\zeta)}{z - \zeta} d\zeta \wedge d\bar{\zeta}},$$

lies in $W^{1,l}(\mathbb{D}_2)$. The conclusion now follows from assertion 3 in Proposition 5.2.1. $\square$

Proof of Theorem 4.2.1. put $r(z)=\alpha(z)\overline{w(z)}/w(z)$ if $w(z)\neq 0$ and $r(z)=0$ if $w(z)=0$ or $z\notin\mathbb{D}$. Then $r\in L^\infty(\mathbb{C})$ and $\|r\|_{L^\infty(\mathbb{D})}\leq\|\alpha\|_{L^\infty(\mathbb{D})}$. Define

$$s(z)=\frac{1}{2\pi i}\iint_{\mathbb{D}}\left(\frac{r(\zeta)}{\zeta-z}+\frac{z\overline{r(\zeta)}}{1-\bar{\zeta}z}\right)d\zeta\wedge d\bar{\zeta}\,,\ \text{for } z\in\mathbb{C}\,. \tag{57}$$

and observe, from Lemma 5.3.1 and assertion 3 in Proposition 5.2.1, that $s\in W^{1,l}_{loc}(\mathbb{C})$ for all $l\in(1,+\infty)$. In particular s is continuous and, since $1/z=\overline{z}$ for $z\in\mathbb{T}$, we see from (57) that $\operatorname{Im}s(z)=0$ there. Also, assertion 4 of Proposition 5.2.1 and Lemma 5.3.1 show that $\bar{\partial}s=r$ in $\mathbb{D}$. Furthermore, a straightforward majorization gives us for $z\in\mathbb{C}$

$$|s(z)|\leq\frac{\|\alpha\|_{L^\infty(\mathbb{D})}}{\pi}\iint_{\mathbb{D}}\left(\frac{1}{|\zeta-z|}+\frac{1}{|\zeta-1/z|}\right)dm\leq 4\|\alpha\|_{L^\infty(\mathbb{D})}\,, \tag{58}$$

thus (23) holds. Next, we put $F=e^{-s}w$ and claim that F is holomorphic in $\mathbb{D}$. Indeed, $F\in L^p(\mathbb{D})$ hence, by Weyl's lemma [34, Thm 24.9], it is enough to check that $\bar{\partial}F=0$ on $\mathbb{D}$ in the sense of distributions. Let $\psi\in\mathcal{D}(\mathbb{D})$ and ψ_n a sequence in $\mathcal{D}(\mathbb{R}^2)_{|\mathbb{D}}$ converging to s in $W^{1,l}(\mathbb{D})$ for some $l>\max(q,2)$. Thus ψ_n converges boundedly to s in $W^{1,q}(\mathbb{D})$ by the Sobolev imbedding theorem and Hölder's inequality. Then, by dominated convergence,

$$\langle\bar{\partial}F,\psi\rangle=-\langle e^{-s}w,\bar{\partial}\psi\rangle=-\lim_n\langle w,e^{-\psi_n}\bar{\partial}\psi\rangle=-\lim_n\langle w,\bar{\partial}(e^{-\psi_n}\psi)+\psi e^{-\psi_n}\bar{\partial}\psi_n\rangle$$

$$=\lim_n\langle\alpha\overline{w},e^{-\psi_n}\psi\rangle-\lim_n\langle w,\psi e^{-\psi_n}\bar{\partial}\psi_n\rangle=\langle e^{-s}(\alpha\overline{w}-w\bar{\partial}s),\psi\rangle=0$$

since $w\bar{\partial}s=wr=\alpha\overline{w}$, where we used in the fourth equality that $e^{-\psi_n}\psi\in\mathcal{D}(\mathbb{D})$. This proves the claim and provides us with (22) where $\operatorname{Im}\operatorname{tr}\ s=0$. Now, by the Sobolev imbedding theorem, s is bounded, and since exp is locally Lipschitz on $\mathbb{C}$ it follows that $w\in W^{1,l}_{loc}(\mathbb{D})$ for all $l\in(1,\infty)$. Finally, by the boundedness of s, it is clear from the definitions that $w\in G^p_\alpha$ if, and only it $F\in H^p$. In this case, it follows from Lemma 5.2.1, point 1, that $w=e^s\,F\in L^{p_1}(\mathbb{D})$ for all $p_1\in[p,2p)$.
To obtain from (22) another factorization $w=e^{s_1}F_1$, where this time $\operatorname{Re}\operatorname{tr}\ s_1=0$, it is enough to change the "+" sign into a "−" one in the definition (57) of s . □

For the proof of Proposition 4.2.1, we need the following version of the Cauchy-Green formula.

Lemma 5.3.2. *When $\psi\in W^{1,p}(\mathbb{D})$, it holds for almost every $z\in\mathbb{D}$ that*

$$\psi(z)=\mathcal{C}\left(tr\ \psi\right)(z)+\frac{1}{2\pi i}\iint_{\mathbb{D}}\frac{\overline{\partial}\psi(\xi)}{\xi-z}d\,\xi\wedge d\,\overline{\xi}. \tag{59}$$

Proof. Note that (59) means $\psi=\mathcal{C}(\mathrm{tr}\ \psi)+T(\bar{\partial}\psi)$. For $\psi\in\mathcal{D}(\mathbb{R}^2)$, this is standard [41, thm. 1.2.1]. In general ψ is the limit in $W^{1,p}(\mathbb{D})$ of a sequence $(\psi_n)_{n\in\mathbb{N}}\in\mathcal{D}(\mathbb{R}^2)|_{\mathbb{D}}$. By continuity of the trace and Proposition 5.2.1, items 1, 4, the conclusion follows from taking a pointwise convergent subsequence of the $L^p(\mathbb{D})$ convergent sequence $T(\bar{\partial}\psi_n)$. □

Proof of Proposition 4.2.1. Let $w\in G^p_\alpha$. By Theorem 4.2.1, we have $w=e^sF$ where $s\in W^{1,l}(\mathbb{D})$, $1<l<\infty$ and $F\in H^p$. As s is continuous on $\overline{\mathbb{D}}$, hence the existence of the non tangential limit (24) and the majorization (27) follow from the corresponding properties of H^p functions [35, thm 3.1]. From Fatou's lemma, we then get the first half of (25), and since $\|F\|_{L^p(\mathbb{T}_r)} \leq \|\mathrm{tr}\ F\|_{L^p(\mathbb{T})}$ for $F \in H^p$ [32, thm 1.5], we obtain by (23)

$$\|w\|_{L^p(\mathbb{T}_r)} \leq e^{2\|s\|_{L^\infty(\mathbb{D})}}\|\mathrm{tr}\ w\|_{L^p(\mathbb{T})} \leq e^{8\|\alpha\|_{L^\infty(\mathbb{D})}}\|\mathrm{tr}\ w\|_{L^p(\mathbb{T})},$$

which yields the second half of (25). Finally, (26) follows from the continuity of s and the corresponding property for H^p-functions [32, thm. 2.6]. This demonstrates assertion 1. Since e^s is continuous and never zero on $\mathbb{D}$, as noticed in Remark 4.2.1, assertion 2 is a consequence of (22) and of the corresponding properties for H^p-functions [32, thms. 2.2, 2.3].

We turn to the proof of assertion 3. Assume first that $w \in L^p(\mathbb{D})$ satisfies $w = \mathcal{C}\varphi + T(\alpha\overline{w})$ for some $\varphi \in L^p(\mathbb{T})$. As $\bar\partial\mathcal{C}\varphi = 0$ on $\mathbb{D}$ because $\mathcal{C}\varphi$ is holomorphic there, we know from Proposition 5.2.1, point 4, that $\bar\partial w = \alpha\bar{w}$ on $\mathbb{D}$. Further, the M. Riesz theorem yields

$$\|\mathcal{C}\varphi\|_{H^p(\mathbb{D})} \leq C_p\|\varphi\|_{L^p(\mathbb{T})}$$

hence $w - T(\alpha\overline{w}) \in H^p$. Lemma 5.2.1 now provides us with the chain of inequalities:

$$\|T(\alpha\overline{w})\|_{L^p(\mathbb{T}_r)} \leq \|T(\alpha\overline{w})\|_{L^\infty(\mathbb{D})} \leq C_p'\|T(\alpha\overline{w})\|_{W^{1,p^*}(\mathbb{D})} \leq C_p''\left(\|w\|_{L^p(\mathbb{D})} + \|\mathcal{C}\varphi\|_{H^p(\mathbb{D})}\right),$$

where we used the Sobolev imbedding theorem. Therefore $w \in G^p_\alpha$ and (30) holds.

Conversely, let $w \in G^p_\alpha$. Then $\bar\partial w = \alpha\bar{w} \in L^p(\mathbb{D})$ and Proposition 5.2.1, assertion 4, tells us that the $L^p(\mathbb{D})$-function $w - T(\alpha\bar{w})$ anihilates $\bar\partial$ in the distributional sense, hence is holomorphic on $\mathbb{D}$ by Weyl's lemma. From Proposition 5.2.1, point 4 again, this entails $w \in W^{1,p}_{loc}(\mathbb{D})$. Appealing to Lemma 5.3.2, with $r\mathbb{D}$ in place of $\mathbb{D}$, we obtain

$$w(z) = \frac{1}{2\pi i}\int_{\mathbb{T}_r}\frac{w(\zeta)}{\zeta - z}d\zeta + T(\alpha\overline{w}\chi_{\mathbb{D}_r})(z)\,, \quad |z| < r < 1,$$

and letting $r \to 1$ we get by dominated convergence, (26), and Proposition 5.2.1, point 4, that

$$w = \mathcal{C}(\mathrm{tr}\ w) + T(\alpha\overline{w}) \quad \text{a. e. in } \mathbb{D}\,. \tag{60}$$

Assertion 3 is now completely proven.

Finally, assume that $w \in G^p_\alpha$ satisfies $\mathrm{Re}\,\mathrm{tr}\ w = 0$. By Theorem 4.2.1 we can write $w = e^sF$, where s is continuous on $\overline{\mathbb{D}}$ and real on $\mathbb{T}$, while $F \in H^p(\mathbb{D})$. Thus $\mathrm{Re}\,\mathrm{tr}\ F$ is zero on $\mathbb{T}$, and by the Poisson representation of H^p-functions it follows that $\mathrm{Re}\,F \equiv 0$ on $\mathbb{D}$ thus F is a pure imaginary constant, say, c [32, Thm 3.1]. Since $w = c\,e^s$, it has zero mean on $\mathbb{T}$ if and only if $c = 0$. When (16) holds, condition (15) likewise implies that $c = 0$. □

5.4 Comparison between Sobolev and Hardy solutions

In this section, we prove Proposition 4.3.3.

Proof of Proposition 4.3.3. We first check that $W^{1,p}(\mathbb{D}) \subset H^p_\nu(\mathbb{D})$ and that (33) holds. Define α through (16). In view of (18) and Proposition 3.2.3.1, point (b), it is enough to check the corresponding property for G^p_α. But if $w \in W^{1,p}(\mathbb{D})$ meets (14), then Lemma 5.3.2 yields $w = \mathcal{C}(\mathrm{tr}\ w) + T(\alpha\overline{w})$, and since $\mathrm{tr}\ w \in W^{1-1/p,p}(\mathbb{T}) \subset L^p(\mathbb{T})$ we get from Proposition 5.2.1, point 1, that $w - T(\alpha\overline{w}) = \mathcal{C}(\mathrm{tr}\ w)$ lies in H^p, implying that $w \in G^p_\alpha$

by Remark 4.2.2. Moreover, it follows from (30) and the trace theorem that $\|w\|_{G^p_\alpha} \leq C_{p,\alpha}\|w\|_{W^{1,p}(\mathbb{D})}$, as desired.

To end the proof of Proposition 4.3.3, we establish the more general fact that, if $f \in W^{1,p}(\mathbb{D})$ has a nontangential limit almost everywhere on $\mathbb{T}$, then this nontangential limit coincides with the trace of f in the Sobolev sense. We provide an argument because we could not locate this "elementary" result in the literature. Note that, when $p > 2$, each $f \in W^{1,p}(\Omega)$ (has a representative which) extends continuouly to $\overline{\Omega}$ by the Sobolev imbedding theorem, so the nontangential limit exists everywhere on $\partial\Omega$ and is in fact an unrestricted limit.
Assume now that $p \leq 2$. Notice that f is the restriction to $\mathbb{D}$ of some $\widetilde{f} \in W^{1,p}(\mathbb{R}^2)$ with compact support [2, Thm 4.26]. By Hölder's inequality $\widetilde{f} \in W^{1,s}(\mathbb{R}^2)$ with some $s \in (1,2)$, hence the non-Lebesgue points of $\widetilde{f}$ have Hausdorff 1-dimension zero [67, Thms 3.3.3, 2.6.16]. In particular, for *each* $r > 0$ we have that $re^{i\theta}$ is a Lebesgue point of $\widetilde{f}$ for a.e. θ. Then, regularizing $\widetilde{f}$ yields a sequence of $\mathcal{D}(\mathbb{R}^2)$-functions converging to $\widetilde{f}$ both in $W^{1,p}(\mathbb{R}^2)$ and pointwise a.e. on every circle $\mathbb{T}_r$ [67, Thm 1.6.1]. Consequently if we put $f_r(\xi) := f(r\xi)$ for $0 < r < 1$ and $\xi \in \mathbb{D}$, we deduce that $\mathrm{tr}\ f_r(\zeta) = f(r\zeta)$ for a.e. $\zeta \in \mathbb{T}$, and since f_r tends to f in $W^{1,p}(\mathbb{D})$ as $r \to 1$ we get that $f(r\zeta) \to \mathrm{tr}\ f(\zeta)$ in $W^{1-1/p,p}(\mathbb{T})$. Finally, as any $W^{1-1/p,p}(\mathbb{T})$-converging sequence has a pointwise a.e. converging subsequence, there is $r_n \to 1$ such that $f(r_n\zeta) \to \mathrm{tr}\ f(\zeta)$ for a.e. $\zeta \in \mathbb{T}$, hence $\mathrm{tr}\ f$ must be the radial (*a fortiori* nontangential) limit of f when the latter exists.

Remark 5.4.1. *Since Sobolev functions can be redefined on a set of zero measure so as to be absolutely continuous on almost every line ([67, Rem. 2.1.5]), it is easy to see (use polar coordinates) that any function in $W^{1,p}(\mathbb{D})$ has a radial limit almost everywhere on $\mathbb{T}$. Note, however, that a function in $W^{1,2}(\mathbb{D})$ may have no nontangential limit at all (see [25]). Note also that a more precise result involving capacity was proven for continuous functions in $W^{1,p}(\mathbb{D})$, see [51, 61].*

5.5 The Dirichlet problem in the class G^p_α

Proof of Theorem 4.4.1.1. Let $g \in H^p$. By assertion 3 of Proposition 4.2.1 and Remark 4.2.2, together with the Cauchy formula for H^p-functions, we see that $w \in L^p(\mathbb{D})$ belongs to G^p_α and satisfies $P_+(\mathrm{tr}\ w) = \mathrm{tr}\ g$ if, and only if $w - T(\alpha\overline{w}) = g$. But since g *a fortiori* lies in $L^p(\mathbb{D})$, there is a unique $w \in L^p(\mathbb{D})$ to meet the latter equation as follows from Proposition 5.2.1, assertion 4. Moreover, by the same assertion, it holds that

$$\|w\|_{L^p(\mathbb{D})} \leq C_{p,\alpha}\|g\|_{L^p(\mathbb{D})} \leq C_{p,\alpha}\|g\|_{H^p},$$

hence (36) holds in view of (30). □

Proof of Theorem 4.4.1.2. For each pair $(\varphi, c) \in L^p_{\mathbb{R}}(\mathbb{T}) \times \mathbb{R}$, set

$$A(\varphi, c) := \left(\mathrm{Re}\left(\mathrm{tr}\ w_{\varphi.c}\right)\ ,\ \mathrm{Im}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\ w_{\varphi.c}(e^{i\theta})\, d\theta\right) \in L^p_{\mathbb{R}}(\mathbb{T}) \times \mathbb{R}\,,$$

where $w_{\varphi,c}$ is the unique function in G^p_α such that $P_+(\mathrm{tr}\ w_{\varphi,c}) = \varphi + i(\mathcal{H}_0\varphi + c)$.

Observe from the M. Riesz theorem that $\varphi \mapsto \varphi + i\mathcal{H}_0\varphi$ is continuous from $L^p_{\mathbb{R}}(\mathbb{T})$ into $\text{tr } H^{p,0} \subset L^p(\mathbb{T})$, hence A is well-defined and continuous from $L^p_{\mathbb{R}}(\mathbb{T}) \times \mathbb{R}$ into itself by (25) and Theorem 4.4.1.1.
Put for simplicity $T_\alpha(w) = T(\alpha\overline{w})$. In view of (35), we have that $(I - T_\alpha)w_{\varphi,c} = g$ where $g \in H^p$ satisfies $\text{tr } g = \varphi + i(\mathcal{H}_0\varphi + c)$, hence $A(\varphi, c) = (\varphi, c) + B(\varphi, c)$ where

$$B(\varphi, c) := \left(\text{Re}\left(\text{tr } T_\alpha(w_{\varphi,c})\right)\ ,\ \text{Im}\frac{1}{2\pi}\int_0^{2\pi} \text{tr } w_{\varphi,c}(e^{i\theta})\,d\theta\ - c\right).$$

Since the G^p_α-norm is finer than the $L^p(\mathbb{D})$-norm, we see from Theorem 4.4.1.1 and Proposition 5.2.1, point 4, that $(\varphi, c) \mapsto T_\alpha(w_{\varphi,c})$ is continuous from $L^p(\mathbb{T}) \times \mathbb{R}$ into $W^{1,p}(\mathbb{D})$, so the first component of B is continuous from $L^p(\mathbb{T}) \times \mathbb{R}$ into $W^{1-1/p,p}(\mathbb{T})$. Therefore it is compact from $L^p(\mathbb{T}) \times \mathbb{R}$ into $L^p(\mathbb{T})$ and, since the second component is $\mathbb{R}$-valued and continuous, B is compact from $L^p(\mathbb{T}) \times \mathbb{R}$ into itself. Moreover, A is injective by Proposition 4.2.1, point 4, consequently it is an isomorphism of $L^p(\mathbb{T}) \times \mathbb{R}$ (see *e.g.* [66, Ch. XVII, prop. 2.3]) thereby establishing the existence and uniqueness part of (a). To establish (37), put $\text{tr } w = \psi + iv$, where $v \in L^p_{\mathbb{R}}(\mathbb{T})$ in view of Proposition 4.2.1. Thanks to Theorem 4.2.1, we can write $\psi + iv = e^{\text{tr } s}F$ where $F \in H^p(\mathbb{D})$ and $\text{tr } s$ is real-valued, $s \in C^{0,\gamma}(\overline{\mathbb{D}})$ for $0 \le \gamma < 1$ (*cf.* Remark 4.2.1). By definition $\text{tr } F = h + i\mathcal{H}_0 h + ib$, where $h \in L^p_{\mathbb{R}}(\mathbb{T})$ and b is a real constant. Thus $v = e^{\text{tr } s}(\mathcal{H}_0 h + b)$ and $\psi = e^{\text{tr } s}h$, which gives us

$$v = e^{\text{tr } s}\mathcal{H}_0(e^{-\text{tr } s}\psi) + be^{\text{tr } s}. \tag{61}$$

Since, v has mean c on $\mathbb{T}$, we get from (23) and the M. Riesz theorem that

$$|b| \le e^{4\|\alpha\|_{L^\infty(\mathbb{D})}}\left(c_p e^{8\|\alpha\|_{L^\infty(\mathbb{D})}}\|\psi\|_{L^p(\mathbb{T})} + |c|\right),$$

where c_p is the norm of $\mathcal{H}_0$ on $L^p(\mathbb{T})$. Plugging this in (61) implies now

$$\|v\|_{L^p(\mathbb{T})} \le e^{8\|\alpha\|_{L^\infty(\mathbb{D})}}\Big(|c| + \left(1 + e^{8\|\alpha\|_{L^\infty(\mathbb{D})}}\right)c_p\|\psi\|_{L^p(\mathbb{T})}\Big)$$

from which (37) follows immediately. This concludes the proof of (a).
Assume next that (16) holds and let $w_1 \in G^p_\alpha$, satisfy $\text{Re}(\text{tr } w_1) = \psi$ a.e. on $\mathbb{T}$. Such a w_1 exists by (a). From the observation made after Proposition 3.2.3.1, we see that the function w defined as

$$w := w_1 - i\sigma^{-1/2}\frac{1}{2\pi}\int_0^{2\pi}\left(\sigma^{1/2}\text{Im}\,\text{tr } w_1\right)(e^{i\theta})\,d\theta$$

lies in G^p_α, and since it readily satisfies (15) we deduce that $w \in G^{p,0}_\alpha$. Clearly w has the same real part as w_1, and by Proposition 4.2.1 point 4 it is the only member of $G^{p,0}_\alpha$ with this property. This settles the existence and uniqueness part in (b).
The reasoning leading to (37) is easily adapted to yield (38), upon trading the mean-equal-to-c condition for (15) and taking (3) into account. This completes the proof. □

5.6 A Fatou theorem for Re H^p_ν

As a preparation for the proof of Theorem 4.4.2.2, we establish the following Hodge type lemma.

Lemma 5.6.1. *Define two subspaces $\mathcal{G}_{0,\infty}$ and $\mathcal{D}_{0,\infty}$ of $C^\infty_\mathbb{R}(\overline{\mathbb{D}})\times C^\infty_\mathbb{R}(\overline{\mathbb{D}})$ by*

$$\mathcal{G}_{0,\infty} := \{\nabla g;\ g\in C^\infty_\mathbb{R}(\overline{\mathbb{D}}),\ tr\ g=0\},\ \mathcal{D}_{0,\infty} := \{(\partial_y h, -\partial_x h)^T;\ h\in C^\infty_\mathbb{R}(\overline{\mathbb{D}}),\ tr\ h=0\}.$$

Then, each $V\in C^\infty_\mathbb{R}(\overline{\mathbb{D}})\times C^\infty_\mathbb{R}(\overline{\mathbb{D}})$ that vanishes on $\mathbb{T}$ can be written uniquely in the form $V=G+D$, where $G\in\mathcal{G}_{0,\infty}$ and $D\in\mathcal{D}_{0,\infty}$. Moreover, it holds for some constant C_p that

$$\|G\|_{L^p(\mathbb{D})}+\|D\|_{L^p(\mathbb{D})}\le C_p\|V\|_{L^p(\mathbb{D})}, \tag{62}$$

where the subscript $L^p(\mathbb{D})$ refers here to the norm of an $\mathbb{R}^2$-valued mapping.

Proof. Recall from (53) the Hodge decomposition $V=G+D$, with $G\in\mathcal{G}_{0,p}$ and $D\in\mathcal{D}_p$, for which (62) is known to hold [43, Thm 10.5.1]. Put $G=(\partial_x g,\partial_y g)^T$ and $D=(\partial_y h,-\partial_x h)^T$ with $g\in W^{1,p}_{0,\mathbb{R}}(\mathbb{D})$ and $h\in W^{1,p}_\mathbb{R}(\mathbb{D})$. Since $V\in C^\infty_\mathbb{R}(\overline{\mathbb{D}})\times C^\infty_\mathbb{R}(\overline{\mathbb{D}})$, the same is true of G and D [43, Sec. 10.5], hence $g,h\in C^\infty_\mathbb{R}(\overline{\mathbb{D}})$ and tr $g=0$. Then, all we have to show is that $h_{|\mathbb{T}}$ is constant, because substracting this constant will produce a new h with vanishing trace on $\mathbb{T}$, as desired.
Now, because tr $g=0$ we deduce that G is normal to $\mathbb{T}$ there, and since $V_{|\mathbb{T}}=0$ it follows that D is also a normal vector field on $\mathbb{T}$. Consequently $x\partial_y h(x,y)-y\partial_x h(x,y)=0$ for $x+iy\in\mathbb{T}$, which means exactly that h is constant on $\mathbb{T}$. □

Proof of Theorem 4.4.2.2. Since $\partial_x(\sigma\partial_x u)=\partial_y(-\sigma\partial_y u)$ by (4), there is a distribution v on $\mathbb{D}$ such that (8) holds [62, Ch. II, Sec. 6, Thm VI]. Then, for $\Phi\in\mathcal{D}_\mathbb{R}(\mathbb{D})$, we obtain

$$\begin{aligned}\langle v,\partial_x\Phi\rangle &= \langle\sigma\partial_y u,\Phi\rangle = -\langle u,\sigma\partial_y\Phi+\Phi\partial_y\sigma\rangle,\\ \langle v,\partial_y\Phi\rangle &= -\langle\sigma\partial_x u,\Phi\rangle = \langle u,\sigma\partial_x\Phi+\Phi\partial_x\sigma\rangle,\end{aligned}$$

which entails by (3), Hölder's inequality, and the Poincaré inequality that

$$|\langle v,\partial_x\Phi\rangle|\le\|u\|_{L^p(\mathbb{D})}C_{p,\sigma}\|\nabla\Phi\|_{L^q(\mathbb{D})}\,,\ |\langle v,\partial_y\Phi\rangle|\le\|u\|_{L^p(\mathbb{D})}C_{p,\sigma}\|\nabla\Phi\|_{L^q(\mathbb{D})}. \tag{63}$$

Next, we observe that any $g\in C^\infty_\mathbb{R}(\overline{\mathbb{D}})$ satisfying tr $g=0$ lies in $W^{1,q}_{0,\mathbb{R}}(\mathbb{D})$, therefore it is the limit in $W^{1,q}_\mathbb{R}(\mathbb{D})$ of some sequence $\Phi_n\in\mathcal{D}_\mathbb{R}(\mathbb{D})$. In particular $\nabla\Phi_n$ converges to ∇g in $L^q(\mathbb{D})$, implying in view of (63) that $(\Phi_1,\Phi_2)\mapsto\langle v,\Phi_1+\Phi_2\rangle$ is a bounded functional on both $\mathcal{G}_{0,\infty}$ and $\mathcal{D}_{0,\infty}$ when endowed with the $L^q(\mathbb{D})$-norm. By Lemma 5.6.1, this functional is $L^q(\mathbb{D})\times L^q(\mathbb{D})$ bounded on the subspace of $C^\infty_\mathbb{R}(\overline{\mathbb{D}})\times C^\infty_\mathbb{R}(\overline{\mathbb{D}})$ comprising those vector fields that vanish on $\mathbb{T}$. Therefore, by density, $(\Phi_1,\Phi_2)\mapsto\langle v,\Phi_1+\Phi_2\rangle$ is a bounded functional on $L^q_\mathbb{R}(\mathbb{D})\times L^q_\mathbb{R}(\mathbb{D})$, so that in fact $v\in L^p_\mathbb{R}(\mathbb{D})$. If we put $f=u+iv\in L^p(\mathbb{D})$ and $\nu=(1-\sigma)/(1+\sigma)\in W^{1,\infty}_\mathbb{R}(\mathbb{D})$, it is now a mechanical consequence of (8) that equation (5) is satisfied in the distributional sense. Defining $w\in L^p(\mathbb{D})$ and $\alpha\in L^\infty(\mathbb{D})$ through (17) and (16), we see from Proposition 3.2.3.1 that w solves (14). Hence Theorem 4.2.1 applies to the effect that w, thus also f and *a fortiori* u, lie in $W^{1,l}_{loc}$ for $l\in(1,\infty)$. Moreover, we may write (22) with $\operatorname{Im}\operatorname{tr}\ s=0$ and, say, $F=a+ib$ a holomorphic function in $\mathbb{D}$. As s lies in $C^{0,\gamma}(\overline{\mathbb{D}})$, for each $\varepsilon>0$ we can pick r_0 such that $|\mathrm{Im}\ \exp(s(z))|<\varepsilon|\exp(s(z))|$ as soon as $r_0<|z|\le 1$, and for such z we deduce from (23) that

$$\operatorname{Re} w(z)\ge e^{-4\|\alpha\|_{L^\infty(\mathbb{D})}}((1-\varepsilon^2)^{1/2}|a(z)|-\varepsilon|b(z)|).$$

Since $b_r=\mathcal{H}_0(a_r)$ (recall that $b_r(z)=b(rz)$, resp. $a_r(z)=a(rz)$, for all $z\in\mathbb{T}$), we obtain from the M. Riesz theorem when $r_0<r<1$ that

$$\|\operatorname{Re} w(z)\|_{L^p(\mathbb{T}_r)}\ge e^{-4\|\alpha\|_{L^\infty(\mathbb{D})}}((1-\varepsilon^2)^{1/2}-\varepsilon c_p)\|a(z)\|_{L^p(\mathbb{T}_r)},$$

and picking ε so small that $((1-\varepsilon^2)^{1/2} - \varepsilon c_p) = C > 0$ we conclude by (17) that

$$\|b\|_{L^p_r(\mathbb{T})} \le c_p \|a\|_{L^p_r(\mathbb{T})} \le C_{p,\sigma,u} \|\mathrm{Re}\, w(z)\|_{L^p(\mathbb{T}_r)} \le C'_{p,\sigma,u} \|u\|_{L^p(\mathbb{T}_r)}, \quad r_0 < r < 1.$$

Consequently, if $\|u\|_{F,p} < \infty$, then $F = a + ib \in H^p$ so that $w \in G^p_\alpha$ by Theorem 4.2.1, whence $f \in H^p_\nu$ by Proposition 3.2.3.1. All the assertions now readily follow from Theorems 4.2.1 and 4.4.2.1. □

5.7 Higher regularity

In order to prove Theorem 4.4.3.1, we shall make use of the following observation:

Lemma 5.7.1. *Let $\varphi \in W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$, and $f \in W^{1,p}(\mathbb{D})$ be the unique solution to* (5) *satisfying* $\mathrm{Re}(tr\ f) = \varphi$ *and* (12)*, cf. Theorem 4.1.1. Then $W = (1-\nu^2)^{1/2}\,\partial f \in L^p(\mathbb{D})$ is a solution to* (14) *with $\alpha = \partial\nu/(1-\nu^2)$, that is to say*

$$\overline{\partial} W = \frac{\partial \nu}{1-\nu^2} \overline{W} \tag{64}$$

in the sense of distributions. Moreover, it holds that

$$\partial f = e^s F \tag{65}$$

where F is holomorphic in $\mathbb{D}$, $s \in C^{0,\gamma}(\overline{\mathbb{D}})$ for every $0 < \gamma < 1$, and for some constant c_κ we have $\|s\|_{L^\infty(\mathbb{D})} \le c_\kappa \|\nu\|_{W^{1,\infty}(\mathbb{D})}$.

Proof. As $\nu \in W^{1,\infty}(\mathbb{D})$, observe that the distributional derivative of $\nu\overline{\partial f} \in L^p(\mathbb{D})$ can be computed according to Leibniz's rule:

$$\partial\left(\nu\overline{\partial f}\right) = \partial\nu\overline{\partial f} + \nu\partial\left(\overline{\partial f}\right), \tag{66}$$

where we emphasize that the second summand in the right-hand side of (66) is to be interpreted as indicated in the footnote before Theorem 4.1.1. Indeed, pick a function $\varphi \in \mathcal{D}(\mathbb{D})$. By definition,

$$\langle \nu\overline{\partial f}, \partial\varphi\rangle = -\langle \nu\overline{f}, \overline{\partial}\partial\varphi\rangle - \langle \overline{\partial}\nu\overline{f}, \partial\varphi\rangle.$$

By the Leibniz rule

$$-\langle \nu\overline{f}, \overline{\partial}\partial\varphi\rangle = -\langle \overline{f}, \nu\overline{\partial}\partial\varphi\rangle = -\langle \overline{f}, \overline{\partial}(\nu\partial\varphi) - \overline{\partial}\nu\partial\varphi\rangle.$$

It follows that

$$\begin{aligned}\langle \nu\overline{\partial f}, \partial\varphi\rangle &= -\langle \overline{f}, \overline{\partial}(\nu\partial\varphi)\rangle = \langle \overline{\partial f}, \nu\partial\varphi\rangle = \langle \overline{\partial f}, \partial(\nu\varphi) - \varphi\partial\nu\rangle \\ &= -\langle \partial\overline{\partial f}, \nu\varphi\rangle - \langle \partial\nu\overline{\partial f}, \varphi\rangle = -\langle \nu\partial\overline{\partial f}, \varphi\rangle - \langle \partial\nu\overline{\partial f}, \varphi\rangle,\end{aligned}$$

which is the desired conclusion.

Setting $G := \partial f$ and applying ∂ to (5), we thus obtain, since ∂ and $\overline{\partial}$ commute, that $\overline{\partial} G = \nu\partial\overline{G} + (\partial\nu)\overline{G}$. As ν is real, conjugating this last equation provides us with an expression for $\partial\overline{G}$, and solving for $\overline{\partial} G$ after substituting back yields

$$\overline{\partial} G = \frac{\nu\overline{\partial}\nu}{1-\nu^2} G + \frac{\partial\nu}{1-\nu^2}\overline{G}$$

from which we deduce that $W = (1-\nu^2)^{1/2}G$ satisfies (64), as $\overline{\partial}W$ can in turn be computed by the chain rule because $(1-\nu^2)^{1/2} \in W^{1,\infty}(\mathbb{D})$. The remaining assertions follow from Theorem 4.2.1 upon setting $\alpha = \partial\nu/(1-\nu^2) \in L^\infty(\mathbb{D})$. □

Proof of Proposition 4.4.3.1. Let $u \in W^{1,p}_{\mathbb{R}}(\mathbb{T}) \subset W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$ and put $v = \mathcal{H}_\nu u$. We must show that $v \in W^{1,p}_{\mathbb{R}}(\mathbb{T})$, and for this we may assume that u has zero mean on $\mathbb{T}$ for adding a constant to u does not affect v. Then, $v = \mathcal{H}_\nu u$ becomes equivalent to $u = -\mathcal{H}_{-\nu}v$ as follows immediately from the fact that f satisfies (5) if, and only if if satisfies a similar equation with ν replaced by $-\nu$. From Theorem 4.1.1 we know that $v \in W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$, so let v_n be a sequence of $C^\infty(\mathbb{T})$-functions converging to v there and set $u_n = -\mathcal{H}_{-\nu}v_n$. Since v_n converges to v in $W^{1-1/p,p}(\mathbb{T})$, we get from Corollary 4.4.2.1 that u_n converges to u there. By the definition of $\mathcal{H}_\nu$, we have that $u_n + iv_n$ is the trace on $\mathbb{T}$ of the solution to (5) in $W^{1,p}(\mathbb{D})$ whose real part on $\mathbb{T}$ is u_n. With a slight abuse of notation, we still designate by u_n, v_n the real and imaginary parts of that solution in $W^{1,p}_{\mathbb{R}}(\mathbb{D})$. By inspection, the generalized Cauchy-Riemann equations (8) do hold with u replaced by u_n and v by v_n, so that u_n, v_n may as well be characterized respectively as the unique solutions in $\mathbb{D}$ to (4), (7) whose traces on $\mathbb{T}$ are our previous u_n and v_n [24]. Now, we know that u_n, v_n lie in $W^{1,p}(\mathbb{D})$; however, since u_n is smooth on $\mathbb{T}$, it follows from [36, Thm 9.15] that in fact $u_n \in W^{2,r}_{\mathbb{R}}(\mathbb{D})$ for all $r \in (1,\infty)$. In view of (8), we deduce that the same is true of v_n as $\sigma \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$. Pick any $\varphi \in C^\infty(\mathbb{T})$ and put $\psi = \mathcal{H}_\nu\varphi$ with ν-conjugate ψ. The $W^{2,r}(\mathbb{D})$ regularity just mentioned allows us by density to apply the divergence formula so as to obtain

$$\int_{\mathbb{T}} v_n \partial_\theta \varphi \, d\theta = -2\pi \int_{\mathbb{T}} v_n \partial_n \psi/\sigma \, d\theta = -2\pi \int_{\mathbb{T}} \partial_n v_n \psi/\sigma \, d\theta = \int_{\mathbb{T}} \partial_\theta u_n \psi \, d\theta,$$

where we used (48). As u_n, v_n converge to u, v in $W^{1-1/p,p}(\mathbb{T})$, we get in the limit, since differentiation is continuous $W^{1-1/p,p}(\mathbb{T}) \to W^{-1/p,p}(\mathbb{T})$, that

$$\int_{\mathbb{T}} v \partial_\theta \varphi \, d\theta = \int_{\mathbb{T}} \partial_\theta u \psi \, d\theta,$$

where $\partial_\theta u$ is to be understood as a member of $W^{-1/p,p}$. However, we have by assumption that in fact $\partial_\theta u \in L^p(\mathbb{T})$, therefore from Hölder's inequality

$$\left| \int_{\mathbb{T}} v \partial_\theta \varphi \, d\theta \right| \leq \|\partial_\theta u\|_{L^p(\mathbb{T})} \|\psi\|_{L^q(\mathbb{T})}.$$

But from Corollary 4.4.2.1 we know that $\|\psi\|_{L^q(\mathbb{T})} \leq C_\nu \|\varphi\|_{L^q(\mathbb{T})}$, so the distribution $\partial_\theta v$ in fact lies in $L^p(\mathbb{T})$ and the conclusion holds. □

Proof of Theorem 4.4.3.1. Write $f = u + iv$ to indicate the real and imaginary parts of f. By Proposition 4.4.3.1, tr $f \in W^{1,p}(\mathbb{T})$. First we shall prove that the left hand sides of (43) and (44) are finite, that is, we show ∂f and $\overline{\partial} f$ satisfy a Hardy condition of order p. To this effect, we consider $w = (f - \nu\bar{f})/\sqrt{1-\nu^2}$ and we establish the equivalent fact that both ∂w and $\bar{\partial} w$ satisfy a Hardy condition of order p. We first notice:

Lemma 5.7.2. *If $p_1 \in [p, 2p)$ then $f, w \in W^{1,p_1}(\mathbb{D})$; in particular $f, w \in C^{0,1-2/p_0}(\overline{\mathbb{D}})$ for some $p_0 > 2$.*

Proof. Note that tr $w \in W^{1,p}(\mathbb{T})$ since tr f does. As we will see in the proof of Proposition 5.2.1 in Appendix A (*cf.* (85)), it entails $\partial(\mathcal{C}\text{tr } w) \in H^p$, and since $\overline{\partial}(\mathcal{C}\text{tr } w) = 0$ we see from the Poincaré inequality and Lemma 5.2.1, point 1, that $(\mathcal{C}\text{tr } w)$ lies in $W^{1,p_1}(\mathbb{D})$. As w, thus $\alpha\overline{w}$ belongs to $L^{p_1}(\mathbb{D})$ by Theorem 4.2.1, we get from Proposition 5.2.1, point 4, that $T_\alpha w \in W^{1,p_1}(\mathbb{D})$. Since

$$w = \mathcal{C}(\text{tr } w) + T_\alpha w,$$

we see that w, therefore also f is in $w \in W^{1,p_1}(\mathbb{D})$. As $2p > 2$, the last assertion now follows from the Sobolev imbedding theorem. □

Back to proof of Theorem 4.4.3.1 we observe that, to prove the finiteness of the left hand sides in (43) and (44), we may as well add a real constant to f. Since the latter is (even Hölder) continuous on $\overline{\mathbb{D}}$ by Lemma 5.7.2, we may thus assume that its real part is larger than a positive constant. Then, the same is true of $w = (f - \nu\overline{f})/\sqrt{1-\nu^2}$, say, $\operatorname{Re} w(z) \geq c_0 > 0$ for $z \in \mathbb{D}$. This results in $\overline{w}/w$ being Hölder continuous of exponent $1 - 2/p_0$ in $\mathbb{D}$. Now, consider the function s introduced in Theorem 4.2.1. Letting $B(z,\varepsilon)$ indicate the ball of center z with radius ε, we gather from (57) that, for a.e. $z \in \mathbb{D}$,

$$\partial s(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \iint_{\mathbb{D}\setminus B(z,\varepsilon)} \frac{r(\zeta)}{(\zeta - z)^2} d\zeta \wedge d\bar{\zeta} + \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{\overline{r(\zeta)}}{(1 - \bar{\zeta} z)^2} d\zeta \wedge d\bar{\zeta}\,, \tag{67}$$

where the function r was defined as $r = \alpha\overline{w}/w$ and the existence of the limit a.e. comes from the existence of the Beurling transform as a singular integral operator of Calderòn-Zygmund type. To evaluate the first integral in (67), we establish a lemma which is best stated in terms of the space $BMOA(\mathbb{D})$, comprised of those H^2-functions whose trace on $\mathbb{T}$ has bounded mean oscillation, see *e.g.* [32, p. 240]. To us, the important fact will be that $BMOA(\mathbb{D}) \subset H^p$ for all $p < \infty$.

Lemma 5.7.3. *There exist a function $b \in L^\infty(\mathbb{D})$ and a function $\psi \in BMOA(\mathbb{D})$ such that, for a.e. $z \in \mathbb{D}$,*

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \iint_{\mathbb{D}\setminus B(z,\varepsilon)} \frac{r(\zeta)}{(\zeta - z)^2} d\zeta \wedge d\bar{\zeta} = b(z) + \psi(z). \tag{68}$$

Proof. We may rewrite the first integral in the right hand side of (67) as

$$\frac{1}{2\pi i} \iint_{\mathbb{D}\setminus B(z,\varepsilon)} \frac{r(\zeta)}{(\zeta - z)^2} d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi i} \iint_{\mathbb{D}\setminus B(z,\varepsilon)} \frac{\alpha(\zeta)((\overline{w}/w)(\zeta) - (\overline{w}/w)(z))}{(\zeta - z)^2} d\zeta \wedge d\bar{\zeta}$$
$$+ \frac{(\overline{w}/w(z))}{2\pi i} \iint_{\mathbb{D}\setminus B(z,\varepsilon)} \frac{\alpha(\zeta)}{(\zeta - z)^2} d\zeta \wedge d\bar{\zeta}\,.$$

Since α is bounded and $\overline{w}/w$ is Hölder continuous of order $1 - 2/p_0$, the first integral in the right hand side is majorized by

$$C_{\nu,w} \iint_{\mathbb{D}} |\zeta - z|^{-1-2/p_0} d\zeta \wedge d\bar{\zeta} \leq C_{\nu,w} \iint_{|\xi| \leq 2} |\xi|^{-1-2/p_0} dm(\xi) < +\infty.$$

As to the second integral, we put for simplicity $\Phi := \log(\sigma^{1/2})$ and we recall from (16) that $\alpha = \overline{\partial}\Phi$, whence by Stoke's theorem

$$\frac{1}{2\pi i} \iint_{\mathbb{D}\setminus B(z,\varepsilon)} \frac{\alpha(\zeta)}{(\zeta - z)^2} d\zeta \wedge d\bar{\zeta} = -\frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\Phi(\zeta) d\zeta}{(\zeta - z)^2} + \frac{1}{2i\pi} \int_{\partial B(z,\varepsilon)} \frac{\Phi(\zeta) d\zeta}{(\zeta - z)^2}. \tag{69}$$

Since

$$\frac{1}{2i\pi}\int_{\partial B(z,\varepsilon)}\frac{\Phi(\zeta)d\zeta}{(\zeta-z)^2}=\frac{1}{2i\pi}\int_{\partial B(z,\varepsilon)}\frac{(\Phi(\zeta)-\Phi(z))d\zeta}{(\zeta-z)^2},$$

and Φ is Lipschitz continuous with constant, say, K (because $\sigma\in W^{1,\infty}(\mathbb{D})$ and in view of (3)) we get

$$\left|\frac{1}{2i\pi}\int_{\partial B(z,\varepsilon)}\frac{(\Phi(\zeta)-\Phi(z))\,d\zeta}{(\zeta-z)^2}\right|\le\frac{K\varepsilon}{2\pi}\int_{\partial B(z,\varepsilon)}\frac{|d\zeta|}{|\zeta-z|^2}=K.$$

Thus the second integral in the right hand side of (69) is uniformly bounded. As to the first, we observe by the Lipschitz character of Φ that $\Phi_{|\mathbb{T}}$ is absolutely continuous on $\mathbb{T}$ with derivative $\varphi:=\partial_\theta\Phi(e^{i\theta})$ which is bounded in modulus by K for a.e. θ. Therefore, integrating by parts, we get

$$\frac{1}{2i\pi}\int_{\mathbb{T}}\frac{\Phi(\zeta)d\zeta}{(\zeta-z)^2}=\frac{1}{2i\pi}\int_{\mathbb{T}}\frac{\varphi(\zeta)d\zeta}{(\zeta-z)}$$

which is the Cauchy integral of a bounded function and therefore belongs $BMOA(\mathbb{D})$ [35, Ch. VI, Cor. 2.5]. □

Lemma 5.7.4. *The function ∂s satisfies a Hardy condition of order l for all $l\in(1,+\infty)$.*

Proof. Let ψ be as in Lemma 5.7.3 and $a(z)$ be the holomorphic integral vanishing at 0 of ψ, namely $\partial a(z)=\psi(z)$ and $\bar\partial a(z)=0$ for $z\in\mathbb{D}$. If we set

$$B(z):=\frac{1}{2\pi i}\iint_{\mathbb{D}}\frac{r(\zeta)}{\zeta-z}d\zeta\wedge d\bar\zeta\,,\ \text{ for } z\in\mathbb{D}\,,$$

we find by (68) and Proposition 5.2.1, point 4, that the bounded function $B-a$ has bounded partial derivatives $\overline{\partial}(B-a)(z)=r(z)$ and $\partial(B-a)(z)=b(z)$, thus it lies in $W^{1,\infty}(\mathbb{D})$ hence it is Lipschitz continuous. But the second summand in the right hand side of (67) is, by our very construction, the derivative of the holomorphic function

$$H(z)=\frac{1}{2\pi i}\iint_{\mathbb{D}}\frac{z\overline{r(\zeta)}}{1-\bar\zeta z}d\zeta\wedge d\bar\zeta\,,\quad z\in\mathbb{D}$$

that vanishes at 0 and whose real part on $\mathbb{T}$ is $\Upsilon=-\mathrm{Re}\,B_{|\mathbb{T}}$, see the discussion after (57). Writing $-B=-(B-a)-a$, we have that $\Upsilon=\Upsilon_1+\Upsilon_2$ where Υ_1 is Lipschitz continuous and thus absolutely continuous with bounded derivative on $\mathbb{T}$, while $\Upsilon_2=-\mathrm{Re}\,(\mathrm{tr}\ a)$. Consequently tr $(H+a)=\Upsilon_1+i\mathcal{H}_0\,(\Upsilon_1)$ where we recall that $\mathcal{H}_0$ denotes the usual conjugation operator. Now, Υ_1 is *a fortiori* in $W^{1,l}(\mathbb{T})$ for all $1<l<\infty$, therefore the same is true of $\mathcal{H}_0\,(\Upsilon_1)$ by Proposition 4.4.3.1 applied with $\nu=0$. Consequently $(H+a)'=H'+\psi$ lies in H^l for all $1<l<\infty$, and finally so does H' since it is the case of $\psi\in BMOA(\mathbb{D})$. Altogether, considering separately the summands in the right-hand side of (67) and recalling Lemma 5.7.3, we have proven that ∂s satisfies a Hardy condition of any order in $(1,+\infty)$. □

Now, let us turn to the holomorphic function $F\in H^p(\mathbb{D})$ in the factorization $w=e^sF$ of Theorem 4.2.1. We claim:

Lemma 5.7.5. *The function $F'\in H^p(\mathbb{D})$.*

Proof. Observe, since $\operatorname{Re} s = 0$ on $\mathbb{T}$, that $|F| = |w|$ there. Moreover, by Lemma 5.7.2 and Remark 4.2.1, $F = e^{-s}w$ is (Hölder) continuous on $\overline{\mathbb{D}}$ and it does not vanish there by our assumption on w. Therefore F can have no inner factor in its inner-outer decomposition [35, Ch. II, Cor. 5.7, Thms 6.2, 6.3] thus it is an outer function:

$$F(z) = \xi_0 \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|w(e^{i\theta})|\, d\theta\right\} \tag{70}$$

with ξ_0 a unimodular constant. As tr $w \in W^{1,p}(\mathbb{T})$ is bounded in modulus from above and below by strictly positive constants, $\log|w(e^{i\theta})|$ also lies in $W^{1,p}(\mathbb{T})$. Hence, in view of (85) in Appendix A below , the derivative of the holomorphic function of z defined by

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|w(e^{i\theta})|\, d\theta = -\frac{1}{2\pi}\int_{\mathbb{T}} \log|w(e^{i\theta})|\, d\theta + \frac{1}{i\pi}\int_{\mathbb{T}} \frac{\log|w(\xi)|}{\xi - z}\, d\xi$$

lies in H^p, and by (70) so does the derivative of F since the latter is bounded. That is, we have proven that $F' \in H^p$. □

Now, since

$$\partial w = e^s \partial s\, F + e^s F',$$

we see by the boundedness of F, s, and the Hardy character of ∂s, F' just established that ∂w satisfies a Hardy condition of order p. Besides, $\bar\partial w = \alpha \bar w$ is bounded, being the product of a $L^\infty(\mathbb{D})$ function and a Hölder continuous one. Thus both ∂w and $\bar\partial w$ satisfy a Hardy condition of order p, and since w is bounded it follows that ∂f and $\overline{\partial} f$ also satisfy a Hardy condition of order p, that is, the left-hand sides of (43) and (44) are finite, as announced.

From this, in view of Lemma 5.7.1, we deduce that $W = (1-\nu)^{1/2}\partial f$ lies in $G^p_{\alpha_1}$ with $\alpha_1 = \partial\nu/(1-\nu^2)$. Clearly α_1 lies in $L^\infty(\mathbb{D})$, hence by (6) and the relation $\overline{\partial} f = \nu\overline{\partial f}$ we deduce from Proposition 4.2.1 that ∂f and $\bar\partial f$ have nontangential limits, say, Φ_1 and $\Phi_2 = \nu_{|\mathbb{T}}\overline{\Phi_1}$ to which $\partial f(re^{i\theta})$ and $\overline{\partial} f(re^{i\theta})$ converge in $L^p(\mathbb{T})$ as $r \to 1$. It only remains for us to establish the explicit expression (46) for tr ∂f, because the latter readily implies that $\|\text{tr } \partial f\|_{L^p(\mathbb{T})} \leq C_\nu \|\text{tr } f\|_{W^{1,p}(\mathbb{T})}$, hence assertions (b) and (c) of Theorem 4.4.3.1 will follow from Proposition 4.2.1 as applied to W, and since we already pointed out that (a) is a rephrasing of Proposition 4.4.3.1 the proof will be complete.

To establish (46), observe by the absolute continuity on a.e. coordinate line characterizing Sobolev functions (choose polar coordinates) that

$$f(re^{i\theta_1}) - f(re^{i\theta_2}) = -\int_{\theta_1}^{\theta_2} \left(r\sin\theta(\partial f + \overline{\partial} f) - ir\cos\theta(\partial f - \overline{\partial} f)\right)(re^{i\theta})\, d\theta$$

for a.e. $r \in (0,1)$ and all θ_1, θ_2. Letting $r \to 1$, we obtain by the continuity of f and the $L^p(\mathbb{T})$-convergence of $\partial f(re^{i\theta})$ and $\overline{\partial} f(re^{i\theta})$ to Φ_1 and Φ_2 that

$$f(e^{i\theta_1}) - f(e^{i\theta_2}) = -\int_{\theta_1}^{\theta_2} (\sin\theta(\Phi_1+\Phi_2) - i\cos\theta(\Phi_1 - \Phi_2))(e^{i\theta})\, d\theta,$$

thereby showing that

$$\partial_\theta f(e^{i\theta}) = ie^{i\theta}\Phi_1(e^{i\theta}) - ie^{-i\theta}\Phi_2(e^{i\theta}) = ie^{i\theta}\Phi_1(e^{i\theta}) - ie^{-i\theta}\nu(e^{i\theta})\overline{\Phi_1(e^{i\theta})}.$$

Conjugating this identity, we obtain since ν is real-valued that

$$\partial_\theta \overline{f(e^{i\theta})} = -ie^{-i\theta}\overline{\Phi_1(e^{i\theta})} + ie^{i\theta}\nu(e^{i\theta})\Phi_1(e^{i\theta}),$$

whence

$$\partial_\theta f(e^{i\theta}) - \nu(e^{i\theta})\partial_\theta\overline{f(e^{i\theta})} = (1-\nu^2)ie^{i\theta}\Phi_1(e^{i\theta}).$$

which is (46).

□

Proof of Corollary 4.4.3.1. Let $f = u + iv \in W^{1,p}(\mathbb{D})$ be the solution to (5) satisfying (12) granted by Theorem 4.1.1. By (47) and (8), f meets

$$\operatorname*{ess\,sup}_{0<r<1} \|\nabla f\|_{L^p(\mathbb{T}_r)} < +\infty. \tag{71}$$

If $W := (1-\nu^2)^{1/2}\partial f$, we saw in the proof of Lemma 5.7.1 that $\overline{\partial} W = \alpha\overline{W}$ for some $\alpha \in L^\infty(\mathbb{D})$. Thus $W \in G^p_\alpha(\mathbb{D})$ by (71), hence it belongs both to $W^{1,l}_{loc}(\mathbb{D})$ for $1 < l < \infty$ and to $L^{p_1}(\mathbb{D})$, for $p_1 \in (p, 2p)$ in view of Theorem 4.2.1. Clearly the same is true of ∂f and $\overline{\partial} f = \nu\overline{\partial f}$, in particular $f \in W^{1,p_2}(\mathbb{D})$ for some $p_2 > 2$, hence is Hölder continuous on $\overline{\mathbb{D}}$. By the Sobolev embedding theorem as applied to its derivatives, f is moreover continuously differentiable in $\mathbb{D}$, so we get for every $re^{i\theta'} \in \mathbb{D}$ that:

$$f(re^{i\theta'}) - f(r) = \int_0^{\theta'} (\partial_\theta f)(re^{i\theta})d\theta \tag{72}$$

where $(\partial_\theta f)(z) = i(z\,\partial f(z) - \bar{z}\nu\overline{\partial f}(z))$. Now, by Proposition 4.2.1 applied to W, the derivatives ∂f and $\overline{\partial} f$ have non tangential limits h and $\nu\bar{h}$ respectively, where $h \in L^p(\mathbb{T})$, and by (26):

$$\lim_{r\to1}\int_0^{2\pi} \left|\partial f(re^{i\theta}) - h(e^{i\theta})\right|^p d\theta = \lim_{r\to1}\int_0^{2\pi} \left|\overline{\partial} f(re^{i\theta}) - \nu\,\bar{h}(e^{i\theta})\right|^p d\theta = 0\,.$$

Passing to the limit as $r \to 1$ in (72) yields that tr f is absolutely continuous on $\mathbb{T}$ with tangential derivative $i(e^{i\theta}h - e^{-i\theta}\nu\bar{h}) \in L^p(\mathbb{T})$, proving that tr $f \in W^{1,p}(\mathbb{T})$.
Since $\partial f \in G^p_\alpha(\mathbb{D})$, Proposition 4.2.1 implies $\mathcal{M}_{\partial f} \in L^p(\mathbb{T})$. The same is true of $\mathcal{M}_{\overline{\partial} f}$ because of (5) and the boundedness of ν, therefore $\mathcal{M}_{\|\nabla f\|} \in L^p(\mathbb{T})$. Moreover, since ∂f and $\overline{\partial f}$ both have non tangential limits in $L^p(\mathbb{T})$, so does ∇f. The same conclusions then hold for $u = \mathrm{Re} f$. In addition, the Green-Riemann formula ensures that

$$0 = \int_{\mathbb{D}} \mathrm{div}(\sigma\nabla u)(x)dx = \int_{\mathbb{T}} \sigma\partial_n \mathrm{tr}\ u\,.$$

This establishes point 1.
As to point 2, let $v \in W^{1,p}_{\mathbb{R}}(\mathbb{T})$ be such that $\partial_\theta v = \sigma g$. By Theorem 4.1.1, there exists $f \in W^{1,p}(\mathbb{D})$ such that (5) holds and Im tr $f = v$ on $\mathbb{T}$. Then, Theorem 4.4.3.1 implies that $u :=$ Re f fulfills all the requirements.
To prove uniqueness up to an additive constant, consider $u \in W^{1,p}_{\mathbb{R}}(\mathbb{D})$ with $\mathrm{div}(\sigma\nabla u) = 0$ in $\mathbb{D}$ and $\partial_n u = 0$ on $\mathbb{T}$. Put $f = u + iv$ for a solution to (5) in $W^{1,p}(\mathbb{D})$ such that $u =$ Re f. By Remark 4.1.2 $\partial_\theta v = \sigma\partial_n u = 0$ on $\mathbb{T}$, which means that tr v is constant, *i.e.* Im tr f is constant. By Theorem 4.1.1, u is constant in $\mathbb{D}$. □

5.8 Proofs of the density results

5.8.1 Density for Sobolev traces

For the proof of Theorem 4.5.1.1, we need two lemmas. Below, for I an open subset of $\mathbb{T}$, the symbol $\langle\cdot,\cdot\rangle_I$ stands for the duality bracket between $W_{\mathbb{R}}^{-1/p,p}(I)$ and $W_{0,\mathbb{R}}^{1-1/q,q}(I)$. We drop the subscript when $I=\mathbb{T}$.

Lemma 5.8.1.1. *For $\varphi\in W_{\mathbb{R}}^{1-1/p,p}(\mathbb{T})$ and $\psi\in W_{\mathbb{R}}^{1-1/q,q}(\mathbb{T})$, one has*

$$\langle\partial_\theta(\mathcal{H}_\nu\varphi),\psi\rangle=\langle\varphi,\partial_\theta(\mathcal{H}_\nu\psi)\rangle. \tag{73}$$

Proof. Assume first that φ and ψ lie in $C^\infty(\mathbb{T})$. Let f and g be the solutions to (5) on $\mathbb{D}$, normalized as in (12), such that Re (tr f) $=\varphi$ and Re (tr g) $=\psi$ respectively. By definition of $\mathcal{H}_\nu$, it holds that Im (tr f) $=\mathcal{H}_\nu\varphi$ and Im (tr g) $=\mathcal{H}_\nu\psi$. As pointed out in the proof of Proposition 4.4.3.1, the functions f and g lie in $W^{2,r}(\mathbb{D})$ for all $r\in(1,\infty)$ by [36, Thm 9.15]. Put $u=$ Re f and $u_1=$ Re g. By the divergence formula

$$0=\int_{\mathbb{D}}(\mathrm{div}(\sigma\nabla u)u_1-\mathrm{div}(\sigma\nabla u_1)u)\,dm=\int_{\mathbb{T}}\sigma\ ((\partial_n u)u_1-(\partial_n u_1)u)\ d\theta,$$

and from (48) we see that $\sigma\partial_n u=\partial_\theta\mathcal{H}_\nu\varphi$ while $\sigma\partial_n u_1=\partial_\theta\mathcal{H}_\nu\psi$. This yields the desired conclusion for smooth φ and ψ. In the general case, pick two sequences $(\varphi_k)_{k\geq1}$, $(\psi_k)_{k\geq1}$ of smooth functions converging respectively to φ in $W^{1-1/p,p}(\mathbb{T})$ and to ψ in $W^{1-1/q,q}(\mathbb{T})$. By Corollary 4.4.2.1, we have that $\mathcal{H}_\nu\varphi_k\to\mathcal{H}_\nu\varphi$ in $W^{1-1/p,p}(\mathbb{T})$ and $\mathcal{H}_\nu\psi_k\to\mathcal{H}_\nu\psi$ in $W^{1-1/q,q}(\mathbb{T})$, therefore $\partial_\theta(\mathcal{H}_\nu\varphi_k)\to\partial_\theta(\mathcal{H}_n u\varphi)$ in $W^{-1/p,p}(\mathbb{T})$ and $\partial_\theta(\mathcal{H}_\nu\psi_k)\to\partial_\theta(\mathcal{H}_\nu\psi)$ in $W^{-1/q,q}(\mathbb{T})$. Identity (73) now follows from the first part of the proof by a limiting argument. □

To proceed with the second lemma, we introduce the following piece of notation that will be of use in the next section as well: whenever $I\subset\mathbb{T}$ and $J=\mathbb{T}\setminus I$ is the complementary subset, then for u_I (resp. u_J) a function on I (resp. J) we put $u_I\vee u_J$ for the concatenated function on $\mathbb{T}$ which is u_I (resp. u_J) on I (resp. J).
Let now I, J be proper open subsets of $\mathbb{T}$ such that $J=\mathbb{T}\setminus\overline{I}\neq\emptyset$. For any function $u_J\in W_{0,\mathbb{R}}^{1-1/p,p}(J)$, we form the concatenated function $0\vee u_J$ and we set

$$A\,u_J=\partial_\theta\big(\mathcal{H}_\nu(0\vee u_J)\big)_{|_I}. \tag{74}$$

Note that $0\vee u_J\in W_{\mathbb{R}}^{1-1/p,p}(\mathbb{T})$, so that $A:W_{0,\mathbb{R}}^{1-1/p,p}(J)\to W_{\mathbb{R}}^{-1/p,p}(I)$ is well-defined and bounded by Corollary 4.4.2.1 and the boundedness of ∂_θ from $W^{1-1/p,p}(I)$ into $W^{-1/p,p}(I)$.

Lemma 5.8.1.2. *The operator A defined in* (74) *has dense range.*

Proof. It is equivalent to show that the adjoint operator $A^*:W_{0,\mathbb{R}}^{1-1/q,q}(I)\to W_{\mathbb{R}}^{-1/q,q}(J)$ is one-to-one. Now, for $u_I\in W_{0,\mathbb{R}}^{1-1/q,q}(I)$ and $u_J\in W_{0,\mathbb{R}}^{1-1/p,p}(J)$, we get by Lemma 5.8.1.1

$$\begin{aligned}\langle A^*u_I,u_J\rangle_J=\langle u_I,Au_J\rangle_I=\langle u_I,\partial_\theta\big(\mathcal{H}_\nu(0\vee u_J)\big)_{|_I}\rangle_I=\langle(u_I\vee0),\partial_\theta\mathcal{H}_\nu(0\vee u_J)\rangle\\=\langle(0\vee u_J),\partial_\theta\mathcal{H}_\nu(u_I\vee0)\rangle=\langle u_J,\partial_\theta\Big(\mathcal{H}_\nu(u_I\vee0)\Big)_{|_J}\rangle_J,\end{aligned}$$

hence $A^* u_I = \partial_\theta\big(\mathcal{H}_\nu(u_I \vee 0)\big)_{|J}$. Thus the relation $A^* u_I = 0$ means that $\mathcal{H}_\nu(u_I \vee 0)$ is constant on each component of J. Let J_0 be such a component and f be the solution of (5) in $W^{1,p}(\mathbb{D})$ such that Re (tr f) $= u_I \vee 0$, normalized as in (12) so that Im (tr f) $= \mathcal{H}_\nu(u_I \vee 0)$. Since $u_I \vee 0$ vanishes on J, there exists $c \in \mathbb{R}$ such that tr $f = ic$ on J_0. Therefore $f \equiv ic$ in $\mathbb{D}$ by Proposition 4.3.3 and Proposition 4.3.1 point (c). In particular Re (tr f) $= 0$ on $\mathbb{T}$, hence $u_I = 0$ as desired. □

Proof of Theorem 4.5.1.1. By definition of an extension set, we can write $I = \cup_j (a_j, b_j)$ where each a_j lies at positive distance from the b_k, therefore also from the a_k for $k \neq j$. This implies there is an open arc I_1 with $\overline{I_1} \neq \mathbb{T}$ such that $\overline{I} \subset I_1$. By the extension property and a smooth partition of unity argument, each function in $W^{1-1/p}(I)$ extends to a function in $W_0^{1-1/p}(I_1)$. Thus, upon trading I for I_1, it is enough to prove that any function in $W_0^{1-1/p,p}(I)$ can be approximated in $W^{1-1/p,p}(I)$ by the trace of a solution of (5).

Let $\varepsilon > 0$ and $\varphi_I \in W_0^{1-1/p,p}(I)$ with $\varphi_I = u_I + iv_I$, where u_I and v_I are real valued. Set $v = (v_I - \mathcal{H}_\nu(u_I \vee 0))_{|I} \in W_{\mathbb{R}}^{1-1/p,p}(I)$. By Lemma 5.8.1.2, there exists $u_J \in W_{0,\mathbb{R}}^{1-1/p,p}(J)$ satisfying

$$\|\partial_\theta v - \partial_\theta \mathcal{H}_\nu(0 \vee u_J)\|_{W^{-1/p,p}(I)} \leq \varepsilon, \tag{75}$$

from which it follows by elementary integration and the Poincaré inequality on I that, for some $c_v \in \mathbb{R}$ and some C_I independent of v,

$$\|v - \mathcal{H}_\nu(0 \vee u_J) - c_v\|_{W^{1-1/p,p}(I)} \leq C_I \varepsilon. \tag{76}$$

Consider now the concatenated function $(u_I \vee u_J) \in W^{1-1/p}(\mathbb{T})$, and define

$$\psi = (u_I \vee u_J) + i\mathcal{H}_\nu(u_I \vee u_J) + ic_v.$$

Then $\psi \in W^{1-1/p,p}(\mathbb{T})$ by Corollary 4.4.2.1, and by construction it is the trace on $\mathbb{T}$ of a solution to (5). Since

$$\psi = (u_I \vee 0) + (0 \vee u_J) + i\mathcal{H}_\nu(u_I \vee 0) + i\mathcal{H}_\nu(0 \vee u_J) + ic_v,$$

it follows from (76) that

$$\|\varphi_I - \psi\|_{W^{1-1/p,p}(I)} \leq C_I \varepsilon.$$

□

5.8.2 Density for Hardy traces

The proof of Theorem 4.5.2.1 resembles that of Theorem 4.5.1.1, but makes use of different operators. Herafter, the symbol $\langle \cdot, \cdot \rangle$ indicates not only the duality between $W_{\mathbb{R}}^{-1/p,p}(\mathbb{T})$ and $W_{0,\mathbb{R}}^{1-1/q,q}(\mathbb{T})$, as in Lemma 5.8.1.1, but also the $L_{\mathbb{R}}^p - L_{\mathbb{R}}^q$ and the $W_{\mathbb{R}}^{1,p} - W_{\mathbb{R}}^{-1,q}$ duality. This should cause no confusion.

First, we will need the following version of Lemma 5.8.1.1:

Lemma 5.8.2.1. *For all $\varphi \in W_{\mathbb{R}}^{1,p}(\mathbb{T})$ and all $\psi \in L_{\mathbb{R}}^q(\mathbb{T})$, one has*

$$\langle \partial_\theta(\mathcal{H}_\nu \varphi), \psi \rangle = \langle \varphi, \partial_\theta(\mathcal{H}_\nu \psi) \rangle.$$

Proof. When φ and ψ are smooth the result is contained in Lemma 5.8.1.1, and the general case then follows by a limiting argument, using the continuiy properties of ∂_θ together with Corollary 4.4.2.1 and Proposition 4.4.3.1. □

From Lemma 5.8.2.1, we deduce a relation between $\mathcal{H}_\nu : L^p_{\mathbb{R}}(\mathbb{T}) \to L^p_{\mathbb{R}}(\mathbb{T})$ and its adjoint $\mathcal{H}^*_\nu : L^q_{\mathbb{R}}(\mathbb{T}) \to L^q_{\mathbb{R}}(\mathbb{T})$.

Lemma 5.8.2.2. *For any function $u \in L^q_{\mathbb{R}}(\mathbb{T})$, it holds that*

$$\mathcal{H}^*_\nu u = -\partial_\theta \mathcal{H}_\nu U,$$

where $U \in W^{1,q}_{\mathbb{R}}(\mathbb{T})$ is any function such that

$$\partial_\theta U = u - \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})d\theta. \tag{77}$$

Proof. Let $u \in L^q_{\mathbb{R}}(\mathbb{T})$, $v \in L^p_{\mathbb{R}}(\mathbb{T})$, and consider $U \in W^{1,q}_{\mathbb{R}}(\mathbb{T})$ such that (77) holds. Then, since $\mathcal{H}_\nu v$ has zero mean on $\mathbb{T}$, we get from Lemma 5.8.2.1

$$\begin{aligned}\langle \mathcal{H}^*_\nu u, v\rangle = \langle u, \mathcal{H}_\nu v\rangle &= \langle u - \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})d\theta \,,\, \mathcal{H}_\nu v\rangle \\ = \langle \partial_\theta U, \mathcal{H}_\nu v\rangle &= -\langle U, \partial_\theta \mathcal{H}_\nu v\rangle = -\langle \partial_\theta \mathcal{H}_\nu U, v\rangle.\end{aligned}$$

□

Remark 5.8.2.1. *If we restrict $\mathcal{H}_\nu$ to the space of $L^q_{\mathbb{R}}(\mathbb{T})$-functions with zero mean, which is mapped into itself by $\mathcal{H}_\nu$, we may summarize the content of Lemma 5.8.2.2 as $\mathcal{H}^*_\nu = -\partial_\theta \mathcal{H}_\nu \partial_\theta^{-1}$.*

Let $I \subset \mathbb{T}$ be as in Theorem 4.5.2.1 and put $J = \mathbb{T} \setminus I$. For $u_J \in L^p_{\mathbb{R}}(J)$, let us define

$$\mathcal{B}\, u_J = \big(\mathcal{H}_\nu(0 \vee u_J)\big)\big|_I .$$

Corollary 5.8.2.1. *The operator $\mathcal{B}$ is bounded from $L^p_{\mathbb{R}}(J)$ to $L^p_{\mathbb{R}}(I)/\mathbb{R}$ and has dense range.*

Note that we consider $\mathcal{B}$ as a mapping from $L^p_{\mathbb{R}}(J)$ into the quotient space $L^p_{\mathbb{R}}(I)/\mathbb{R}$ rather than $L^p_{\mathbb{R}}(I)$. That $\mathcal{B}$ has dense range means: given $v \in L^p_{\mathbb{R}}(I)$ and $\varepsilon > 0$, there exist $u_J \in L^p_{\mathbb{R}}(J)$ and $c_v \in \mathbb{R}$ such that

$$\|v - \mathcal{H}_\nu(0 \vee u_J) - c_v\|_{L^p(I)} \le \varepsilon. \tag{78}$$

Proof. Clearly $\mathcal{B}$ is well-defined and bounded by Corollary 4.4.2.1. To prove it has dense range, it is enough to check that $\mathcal{B}^* : L^{q,0}_{\mathbb{R}}(I) \to L^q(J)$ is one-to-one; here, by $L^{q,0}_{\mathbb{R}}(I)$, we mean the subspace of $L^q_{\mathbb{R}}(I)$ comprised of functions with zero mean. For $\varphi_I \in L^{q,0}_{\mathbb{R}}(I)$, we deduce from Lemma 5.8.2.2 that

$$\mathcal{B}^*\varphi_I = -\left(\partial_\theta \mathcal{H}_\nu \Psi\right)_{|J}, \tag{79}$$

where $\Psi \in W^{1,q}_{\mathbb{R}}(\mathbb{T})$ is such that

$$\partial_\theta \Psi = \varphi_I \vee 0. \tag{80}$$

Consider the solution $f \in W^{1,p}(\mathbb{D})$ to (5) satisfying $\mathrm{Re}\,(\mathrm{tr}\ f) = \Psi$ on $\mathbb{T}$, normalized so that (12) holds. By (80) it holds that $\partial_\theta \Psi = 0$ a.e. on J, and if $\mathcal{B}^* \varphi_I = 0$ we get in addition from (79) that $\partial_\theta \mathcal{H}_\nu \Psi = 0$ a.e. on J. As $\mathrm{tr}\ f = \Psi + i\mathcal{H}_\nu \Psi$, it entails altogether that $\partial_\theta(\mathrm{tr}\ f) = 0$ a.e. on J. But from Theorem 4.4.3.1 point (c) we know that ∂f admits a non tangential limit a.e. on $\mathbb{T}$, and by (46) we now see that $\mathrm{tr}\ \partial f = 0$ a.e. on J. But Theorem 4.4.3.1 point (b) and Lemma 5.7.1 equation (65) together imply that $\partial f = e^s F$, where s is continuous on $\overline{\mathbb{D}}$ while $F \in H^p$. Necessarily then, $\mathrm{tr}\ F = 0$ on J which has positive measure, hence $F \equiv 0$ implying that $\partial f = \overline{\partial} f \equiv 0$. Therefore f is constant, in particular $0 = \partial_\theta \mathrm{Re}(\mathrm{tr}\ f) = \partial_\theta \Psi = \varphi_I \vee 0$ on $\mathbb{T}$. Therefore $\varphi_I \equiv 0$ thus $\mathcal{B}^*$ is injective, as desired. □

Proof of Theorem 4.5.2.1. Let $\varepsilon > 0$ and $\varphi_I \in L^p(I)$ with $\varphi_I = u_I + iv_I$ where $u_I, v_I \in L^p_{\mathbb{R}}(I)$. Set $v = v_I - \mathcal{H}_\nu(u_I \vee 0)$ and observe from (78) that

$$\|v - \mathcal{H}_\nu(0 \vee u_J) - c_v\|_{L^p(I)} \leq \varepsilon. \tag{81}$$

for some $u_J \in L^p_{\mathbb{R}}(J)$ and $c_v \in \mathbb{R}$. Define

$$\psi = (u_I \vee u_J) + i\mathcal{H}_\nu(u_I \vee u_J) + ic_v.$$

By construction ψ is the trace on $\mathbb{T}$ of a H^p_ν-function, and since

$$\psi = (u_I \vee 0) + (0 \vee u_J) + i\mathcal{H}_\nu(u_I \vee 0) + i\mathcal{H}_\nu(0 \vee u_J) + ic_v,$$

it follows from (81) that

$$\|\varphi_I - \psi\|_{L^p(I)} \leq \varepsilon.$$

The proof is now complete. □

5.9 A characterization of $(\mathrm{tr}\ H^p_\nu)^\perp$

The expression for $\mathcal{H}^*_\nu$ obtained in Lemma 5.8.2.2 enables us to characterize the orthogonal space $(\mathrm{tr}\ H^p_\nu)^\perp$ of $\mathrm{tr}\ H^p_\nu$ for the duality product (49), hereafter denoted by $\langle ., . \rangle$.

Proof of Proposition 4.6.1. Let $\varphi = \varphi_1 + i\varphi_2$, with $\varphi_k \in L^q_{\mathbb{R}}(\mathbb{T})$. In view of Theorem 4.4.2.1, we get that $\varphi \in (\mathrm{tr}\ H^p_\nu)^\perp$ if, and only if

$$\mathrm{Re}\,\langle \varphi_1 + i\varphi_2, u + i\mathcal{H}_\nu u + ic \rangle = 0\,, \quad u \in L^p_{\mathbb{R}}(\mathbb{T}),\ c \in \mathbb{R}\,. \tag{82}$$

Picking for u an arbitrary constant shows that φ_1 and φ_2 have zero mean on $\mathbb{T}$, hence there are $\Phi_1, \Phi_2 \in W^{1,q}_{\mathbb{R}}(\mathbb{T})$ such that $\Phi := \Phi_1 + i\Phi_2$ satisfies $\partial_\theta \Phi = \varphi$; we may impose in addition that Φ itself has zero mean. Now, from (82) we get for every $u \in L^p_{\mathbb{R}}(\mathbb{T})$ that

$$0 = \mathrm{Re}\,\langle u + i\mathcal{H}_\nu u, \varphi_1 + i\varphi_2 \rangle = \langle u, \varphi_1 \rangle - \langle u, \mathcal{H}^*_\nu \varphi_2 \rangle,$$

which means $\varphi_1 = \mathcal{H}^*_\nu \varphi_2$ or else $\Phi_1 = -\mathcal{H}_\nu \Phi_2$, by elementary integration and using Lemma 5.8.2.2. Therefore $\Phi = i\,(\Phi_2 + i\,\mathcal{H}_\nu \Phi_2)$ lies in $i\mathrm{tr}\ H^q_\nu = \mathrm{tr}\ H^q_{-\nu}$, and since $\partial_\theta \Phi = \varphi$ we conclude that

$$(\mathrm{tr}\ H^p_\nu)^\perp \subset \partial_\theta \left(\mathrm{tr}\ H^q_{-\nu} \cap W^{1,q}(\mathbb{T})\right)\,.$$

Conversely let $\varphi = \partial_\theta \Phi$ for some $\Phi \in (\text{tr } H^q_{-\nu} \cap W^{1,q}(\mathbb{T}))$. We can write $\Phi = \gamma + i\,\mathcal{H}_{-\nu}\,\gamma$ for some $\gamma \in W^{1,q}_{\mathbb{R}}(\mathbb{T})$, and from Lemma 5.8.2.2 applied with $-\nu$ rather than ν we obtain for $u \in L^p_{\mathbb{R}}(\mathbb{T})$

$$\begin{aligned}\text{Re}\,\langle \varphi, u + i\mathcal{H}_\nu u\rangle &= \langle \partial_\theta\,\gamma, u\rangle - \langle \partial_\theta\,(\mathcal{H}_{-\nu}\,\gamma), \mathcal{H}_\nu\,u\rangle \\ = \langle \partial_\theta\,\gamma, u\rangle + \langle \mathcal{H}^*_{-\nu}(\partial_\theta\,\gamma), \mathcal{H}_\nu\,u\rangle &= \langle \partial_\theta\,\gamma, u\rangle + \langle \partial_\theta\,\gamma, \mathcal{H}_{-\nu}(\mathcal{H}_\nu\,u)\rangle \\ &= \langle \partial_\theta\,\gamma, \int_0^{2\pi} u d\theta\rangle = 0\,,\end{aligned}$$

since $\mathcal{H}_{-\nu}(\mathcal{H}_\nu\,u) = -u + \int_0^{2\pi} u d\theta$ as follows immediately from the fact that $iH^p_\nu = H^p_{-\nu}$. □

Remark 5.9.1. *In order to extend the definition of $\mathcal{H}_\nu$ to complex valued functions, it is natural to set (cf. [7, 8])*

$$\mathcal{H}_\nu(i\,u) = i\,\mathcal{H}_{-\nu}(u)\,,$$

for we then have

$$i\,u + i\,\mathcal{H}_\nu(i\,u) = i\,(u + i\,\mathcal{H}_{-\nu}(u))\,,$$

which is indeed a solution to (5)*. With this definition, we recap Proposition 4.6.1 by saying that $\varphi \in L^q(\mathbb{T})$ belongs to (tr $H^p_\nu)^\perp$ if and only if $\varphi = \partial_\theta \Phi$, where $\Phi \in W^{1,q}(\mathbb{T})$ satisfies*

$$\mathcal{H}_{-\nu}\Phi = -i\,\Phi,. \tag{83}$$

Note that if $\nu = 0$ then (83) *characterizes traces of holomorphic functions with zero mean. In this case* (82) *follows readily from the Cauchy formula.*

6 Hardy spaces over Dini-smooth domains

In this final section, we indicate how the spaces H^p_ν that we studied on the disk can be defined more generally on Dini-smooth domains.

Recall that a function h is called *Dini-continuous* if $\int_0^\varepsilon (\omega_h(t)/t)dt < +\infty$ for some, hence any $\varepsilon > 0$, where ω_h is the modulus of continuity of h. A function is said to be *Dini-smooth* if it has Dini-continuous derivative. A bounded planar domain in $\mathbb{C}$ is termed Dini-smooth if its boundary is a Jordan curve with nonsingular Dini-smooth parametrization. Such domains Ω have the property that any conformal map from $\mathbb{D}$ onto Ω extends continuously from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ together with its derivative, in such a way that the latter is never zero [58, thm 3.5], and that is why we are able to generalize our previous results to this class of domains.

Let $\Omega \subset \mathbb{C}$ be a simply connected Dini-smooth domain, as defined in the introduction, and assume that $\nu \in W^{1,\infty}(\Omega)$ with $\|\nu\|_{L^\infty(\Omega)} \leq \kappa < 1$. Any conformal transformation ψ from $\mathbb{D}$ onto Ω has a $C^1(\overline{\mathbb{D}})$ conformal extension onto $\overline{\Omega}$ that we still denote by ψ [58, thm 3.5]. Introduce the Hardy classes $H^p_\nu(\Omega)$ as the space of functions f on Ω such that $f \circ \psi \in H^p_{\nu\circ\psi}(\mathbb{D})$. A straightforward computation [3, Ch. 1, C] shows that this class does not depend on the particular choice of ψ and consists of distributional solutions to (5). Note that $\nu \circ \psi \in W^{1,\infty}(\mathbb{D})$ with $\|\nu \circ \psi\|_{L^\infty(\mathbb{D})} \leq \kappa < 1$. Similarly, the class $G^p_\alpha(\Omega)$ consists of those functions g on Ω such that $g \circ \psi \in G^p_{\alpha\circ\psi}(\mathbb{D})$.
As in the classical case of holomorphic Hardy spaces over simply connected domains [32], it appears that $w \in G^p_\alpha(\Omega)$ if and only if it is a solution to (14) such that $|w|^p$ has a harmonic

majorant (we insist that harmonic is understood here in the usual sense). Indeed, it is enough to check this property in $\mathbb{D}$, since it is preserved by composition with conformal maps. Then, it is a consequence of Theorem 4.2.1 that functions in $G^p_\alpha(\mathbb{D})$ possess this property. For the converse, observe that any solution w to (14) can be factorized as $w = e^s F$, where $s \in C(\overline{\mathbb{D}})$ and F is holomorphic in $\mathbb{D}$. Since $|F|^p = e^{-p\mathrm{Re}\, s}\, |w|^p$, it admits a harmonic majorant, and therefore it belongs to $H^p(\mathbb{D})$, which immediately yields that $w \in G^p_\alpha(\mathbb{D})$.

Another possibility to generalize H^p_ν to a Dini-smooth simply connected domain Ω is to parallel the definition of the so-called Smirnov classes [32, Ch. 10]. Namely, one requires the uniform boundedness of the L^p norm on a sequence of rectifiable contours eventually encompassing every compact subset of Ω. For Dini-smooth domains, the two generalizations turn out to be equivalent.

The results of Sections 4.2, 4.3, 4.4, and 4.5 remain valid in this framework, as can be seen easily by composing with ψ and appealing to the regularity of ψ^{-1}.

7 Conclusion

This paper took a few steps towards a theory of two-dimensional Hardy spaces for the conjugate Beltrami equation on simply connected domains. Conspicuously missing is a factorization theory, whose starting point should be a characterization of those pairs (s, F) for which (22) holds. Also, we did not pursue the solution to the extremal problems stated in Theorem 4.6.1, points $(ii)-(iii)$. Finally, the case of multiply connected domains, that motivated in part the present study (*cf.* the introduction), was not touched upon. It is to be hoped that suitable deepenings of the present techniques will enable one to approach such issues.

A Appendix. Proof of technical results

We establish here Proposition 5.2.1 and Lemma 5.2.1.

Proof of Proposition 5.2.1: The boundedness of $\mathcal{C}$ from $L^p(\mathbb{T})$ onto $H^p(\mathbb{D})$ follows from the M. Riesz theorem, mentioned already. To establish the second half of 1, let $\varphi \in L^p(\mathbb{T})$. From the definition of the H^p-norm and the M. Riesz theorem, we get for each $r \in (0,1)$ and every $\epsilon > 0$ that

$$\|\mathcal{C}\varphi\|_{L^p(\mathbb{T}_r)} \leq C_p\, \|\varphi\|_{L^p(\mathbb{T})} \leq \frac{C_p}{(1-r)^\epsilon} \|\varphi\|_{L^p(\mathbb{T})}\,,$$

where the last inequality is trivial. It then follows from [32, thm 5.5] that

$$\|\partial\mathcal{C}\varphi\|_{L^p(\mathbb{T}_r)} \leq \frac{C_{\epsilon,p}}{(1-r)^{1+\epsilon}} \|\varphi\|_{L^p(\mathbb{T})}\,. \tag{84}$$

If moreover $\varphi \in W^{1,p}(\mathbb{T})$, integrating by parts the Cauchy formula gives us

$$\partial\mathcal{C}\varphi(z) = \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{\varphi(\xi)}{(\xi - z)^2}\, d\xi = \frac{1}{i}\int_{\mathbb{T}} \frac{\partial_t\varphi(\xi)}{\xi(\xi - z)}\, d\xi,$$

from which it follows by the M. Riesz theorem again that

$$\|\partial \mathcal{C}\varphi\|_{L^p(\mathbb{T}_r)} \leq C_p \|\varphi\|_{W^{1,p}(\mathbb{T})}. \tag{85}$$

Introduce the operators $A_r^1 : W^{1,p}(\mathbb{T}) \to L^p(\mathbb{T}_r)$ and $A_r^0 : L^p(\mathbb{T}) \to L^p(\mathbb{T}_r)$, where in both cases $A_r^i\varphi = (\partial\mathcal{C}\varphi)_{|\mathbb{T}_r}$. We gather from (85) and (84) that

$$\|A_r^1\| \leq C_p \quad \text{while} \quad \|A_r^0\| \leq \frac{C_{\epsilon,p}}{(1-r)^{1+\epsilon}}.$$

Since $W^{1-1/p,p}(\mathbb{T}) = [W^{1,p}(\mathbb{T}), L^p(\mathbb{T})]_{1/p}$, interpolating between A_r^1 and A_r^0 yields

$$\|\partial\mathcal{C}\varphi\|_{L^p(\mathbb{T}_r)} \leq \frac{C'_{\epsilon,p}}{(1-r)^{(1+\epsilon)/p}} \|\varphi\|_{W^{1-1/p,p}(\mathbb{T})}.$$

Choosing ϵ so small that $p > 1+\epsilon$, the above right hand side is integrable w.r.t. $r \in (0,1)$, so by Fubini's theorem $\partial\mathcal{C}\varphi \in L^p(\mathbb{D})$ as soon as $\varphi \in W^{1-1/p,p}(\mathbb{T})$. Since $\bar\partial\mathcal{C}\varphi = 0$, this establishes 1.

Assertion 2 is a consequence of the fact that S is a $L^2(\mathbb{C})$-isometric Calderón-Zygmund operator, see [3, Ch. V, Sec. D] or [64, Ch. II, Thm 3].

Next, observe that if $K \subset \mathbb{C}$ and $w \in L^p(\mathbb{C})$, we have for $z \in K$

$$\breve{T}w(z) = \frac{1}{\pi}\left((\chi_{\mathbb{D}}w) * g_K\right)(z), \quad \text{with} \quad g_K(\xi) := \frac{\chi_{K+\mathbb{D}}(\xi)}{\xi},$$

where χ_E denotes the characteristic function of a set E. Clearly $g_K \in L^1(\mathbb{C})$ when K is compact, showing that $\breve{T}$ is bounded from $L^p(\mathbb{C})$ into $L^p_{loc}(\mathbb{C})$. We claim that $\bar\partial\breve{T}w = \chi_{\mathbb{D}}w$ and $\partial\breve{T}w = S(\chi_{\mathbb{D}}w)$ in the sense of distributions. When w is C^2-smooth and compactly supported in $\mathbb{D}$, whence $\chi_{\mathbb{D}}w = w$, this is a simple computation [3, Ch. V, Lem. 2]. In the general case, pick a sequence of functions $v_n \in \mathcal{D}(\mathbb{D})$ converging to $\chi_{\mathbb{D}}w$ in $L^p(\mathbb{C})$. By what precedes $\breve{T}v_n$ converges to $\breve{T}(\chi_{\mathbb{D}}w) = \breve{T}w$ in $L^p_{loc}(\mathbb{C})$, hence as distributions

$$\bar\partial(\breve{T}w) = \lim_n \bar\partial(\breve{T}v_n) = \lim_n v_n = \chi_{\mathbb{D}}w, \tag{86}$$

where the last limit holds in $L^p(\mathbb{C})$ thus *a fortiori* in the distributional sense. Using the $L^p(\mathbb{C})$ boundedness of S, a similar argument yields

$$\partial(\breve{T}w) = \lim_n \partial(\breve{T}v_n) = \lim_n S(v_n) = S(\chi_{\mathbb{D}}w), \tag{87}$$

proving the claim. Since $\left\|\partial\breve{T}w\right\|_{L^p(\mathbb{C})} = \|S(\chi_{\mathbb{D}}w)\|_{L^p(\mathbb{C})} \leq C\,\|w\|_{L^p(\mathbb{D})}$, assertion 3 holds.

As to assertion 4, the boundedness of T from $L^p(\mathbb{D})$ to $W^{1,p}(\mathbb{D})$ follows from 3 and the identity $(\breve{T}\breve{w})_{|\mathbb{D}} = Tw$, an so do the relations $\bar\partial Tw = w$ and $\partial Tw = (S\breve{w})_{|\mathbb{D}}$ in view of (86) and (87). The compactness of $T : L^p(\mathbb{D}) \to L^p(\mathbb{D})$ then follows from the compactness of the injection $W^{1,p}(\mathbb{D}) \to L^p(\mathbb{D})$ [2, Ch. VI, Thm 6.2].

Finally, set $T_\alpha w = T(\alpha\overline{w})$. By a theorem of F. Riesz, to see that $I - T_\alpha$ is an isomorphism of $L^p(\mathbb{D})$, it is enough since T_α is compact to check that $I - T_\alpha$ is one-to-one. Let $w \in L^p(\mathbb{D})$ be such that $(I - T_\alpha)w = 0$ and set $u = \breve{T}(\breve{\alpha\overline{w}}) \in W^{1,p}_{loc}(\mathbb{C})$. Observe there is $p_1 > 2$ such that $u \in L^{p_1}_{loc}(\mathbb{C})$. Indeed, by the Sobolev imbedding theorem, if $p < 2$ then $W^{1,p}_{loc}(\mathbb{C})$

is embedded in $L^{p^*}_{loc}(\mathbb{C})$, with $p^* = \frac{2p}{2-p}$, whereas if $p \geq 2$ then $W^{1,p}_{loc}(\mathbb{C})$ is embedded in $L^{\lambda}_{loc}(\mathbb{C})$ for every $\lambda \in (2,\infty)$.
Now, since $w = T_\alpha w$, we have $u = w$ in $\mathbb{D}$ and so $w \in L^{p_1}(\mathbb{D})$ hence $\breve{\alpha}\overline{w} \in L^{p_1}(\mathbb{C})$. Thus, by assertion 3, we get that $u \in W^{1,p_1}_{loc}(\mathbb{C})$. Moreover, from (86) and since $u = w$ in $\mathbb{D}$, it holds in the sense of distributions that

$$\bar{\partial} u = \breve{\alpha}\overline{w} = \breve{\alpha}\,\overline{u} \quad \text{a.e. in } \mathbb{C}. \tag{88}$$

In addition, $u(z)$ clearly goes to 0 when $|z|$ goes to $+\infty$. It now follows from the generalized Liouville theorem [7, Prop. 3.3] that $u = 0$, therefore $w = 0$. □

Proof of Lemma 5.2.1: It is a result by Hardy and Littlewood [32, thm 5.9] that for any $g \in H^p$

$$\|g\|_{L^{p_1}(\mathbb{T}_r)} \leq C_p \|g\|_{L^p(\mathbb{T}_r)} (1-r)^{1/p_1 - 1/p}\,,\ 0 < r < 1\,,\quad p \leq p_1 \leq \infty\,. \tag{89}$$

Taking $p_1 \in [p, 2p)$ and raising (89) to the power p_1, we obtain upon integrating with respect to r

$$\|g\|^{p_1}_{L^{p_1}(\mathbb{D})} \leq C_{p,p_1} \|\mathrm{tr}\ g\|^{p_1}_{L^p(\mathbb{T})}$$

thereby proving 1. For the rest of the proof, we fix $p_1 \in (2, 2p)$.
Assume now that $w - T(\alpha\overline{w}) = g \in H^p$. By Proposition 5.2.1, point 4, we get $T(\alpha\overline{w}) \in W^{1,p}(\mathbb{D})$. Moreover, as already stressed in the proof of that proposition, we know from the Sobolev imbedding theorem that $W^{1,p}(\mathbb{D}) \subset L^{p_2}(\mathbb{D})$ for some $p_2 > 2$. Thus, by point 1 and Hölder's inequality, $w = g + T(\alpha\overline{w})$ belongs to $L^{p^*}(\mathbb{D})$ with $p^* = \min\{p_1, p_2\} > 2$, and consequently $T(\alpha\overline{w}) \in W^{1,p^*}(\mathbb{D}) \subset C^{0,1-2/p^*}(\overline{\mathbb{D}})$ by the Sobolev imbedding theorem again. Using Proposition 5.2.1, point 3, we establish similarly that $\breve{T}\left(\breve{\alpha}\overline{w}\right) \in W^{1,p^*}_{loc}(\mathbb{C}) \subset C^{0,1-2/p^*}_{loc}(\mathbb{C})$. Recaping what we just did, we obtain

$$\|T(\alpha\overline{w})\|_{W^{1,p^*}(\mathbb{D})} \leq c_{p,\alpha} \|w\|_{L^{p^*}(\mathbb{D})} = c_{p,\alpha} \|g + T(\alpha\overline{w})\|_{L^{p^*}(\mathbb{D})} \leq c'_{p,\alpha} \left(\|T(\alpha\overline{w})\|_{L^{p_2}(\mathbb{D})} + \|g\|_{L^{p_1}(\mathbb{D})} \right)$$

$$\leq c''_{p\alpha} \left(\|T(\alpha\overline{w})\|_{W^{1,p}(\mathbb{D})} + \|g\|_{H^p(\mathbb{D})} \right) \leq c'''_{p,\alpha} \left(\|w\|_{L^p(\mathbb{D})} + \|g\|_{H^p(\mathbb{D})} \right)$$

which is (56). □

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